Third-order cosmological perturbations of zero-pressure multi-component fluids: Pure general relativistic nonlinear effects

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Present expansion stage of the universe is believed to be mainly governed by the cosmological constant, collisionless dark matter and baryonic matter. The latter two components are often modeled as zero-pressure fluids. In our previous work we have shown that to the second-order cosmological perturbations, the relativistic equations of the zero-pressure, irrotational, multi-component fluids in a spatially near flat background effectively coincide with the Newtonian equations. As the Newtonian equations only have quadratic order nonlinearity, it is practically interesting to derive the potential third-order perturbation terms in general relativistic treatment which correspond to pure general relativistic corrections. In our previous work we have shown that even in a single component fluid there exists substantial amount of pure relativistic third-order correction terms. We have, however, shown that those correction terms are independent of the horizon scale, and are quite small ($\sim 5 \times 10^{-5}$ smaller compared with the relativistic/Newtonian second-order terms) due to the weak level anisotropy of the cosmic microwave background radiation. Here, we present pure general relativistic correction terms appearing in the third-order perturbations of the multi-component zero-pressure fluids. As a result we show that, as in a single component situation, the third-order correction terms are quite small ($\sim 5 \times 10^{-5}$ smaller) in the context of the evolution of our universe. Still, there do exist pure general relativistic correction terms in third-order perturbations which could potentially become important in future development of precision cosmology. We include the cosmological constant in all our analyses.

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I.   INTRODUCTION

Recently, we have been presenting a series of work based on our theoretical study of relativistic nonlinear cosmological perturbations [1, 2, 3, 4]. We have shown that to the second-order perturbations, general relativistic equations of a zero-pressure, irrotational fluid in a spatially flat background have exact Newtonian correspondence except for the presence of the gravitational wave contributions [1, 2]. In an accompanying paper [5] we have relaxed all the assumptions we made in the second-order perturbations in [1, 2], and have derived pure general relativistic effects from the pressures, rotation, spatial curvature, and multi-component. In that work, we have shown that except for the multi-component situation, relaxing any of the other three assumptions leads to pure general relativistic correction terms appearing in the second order. Pressures are intrinsically general relativistic even in the background and the linear-order perturbations. The presence of background curvature leads to first non-vanishing relativistic correction terms appearing in the second order. The rotational perturbations generally lead to relativistic correction terms which become negligible in the small-scale (sub-horizon scale) limit, thus having relativistic/Newtonian correspondence in that limit. In the case of zero-pressure, irrotational multi-component fluids in a flat background, effectively we have exact relativistic/Newtonian correspondence even in the multi-component situation; this will be summarized in the later section of this work.

The relativistic/Newtonian correspondence in the background world model was known in the zero-pressure medium by the work of Friedmann in 1922 [6] in the context of Einstein's gravity, and by the work of Milne and McCrea in 1934 [7] in the context of Newton's gravity; the latter Newtonian derivation is later known to be a guided one by the already derived Einstein gravity result [8]. In the case of linear perturbations, the relativistic/Newtonian correspondence was also known in the zero-pressure medium by the work of Lifshitz in 1946 [9] in the context of Einstein's gravity, and by the work of Bonnor in 1957 [10] in the context of Newton's gravity. The fully nonlinear perturbation equations in the context of Newtonian cosmology in a zero-pressure medium were known in a textbook by Peebles in 1980 [11]. The Einstein's gravity counterpart of the nonlinearly perturbed cosmological medium, especially the continued relativistic/Newtonian correspondence even to the second order, was first shown only recently in our works in [1, 2].

Meanwhile, in [4], we derived pure general relativistic correction terms appearing in the third-order single component, zero-pressure, irrotational fluid in a flat background. Thus, now it is a natural step to find out the potential third-order pure general relativistic correction terms appearing in the multi-component, zero-pressure, irrotational fluids in a flat background. As the Newtonian system has only quadratic nonlinearity even in the multi-component situation, see Sec. II of [5], any nonvanishing third-order terms can be regarded as pure general relativistic corrections. The situation is also practically important because current stage of the universe is supposed to be dominated by two zero-pressure components and the cosmological constant. We will include the effect of cosmological constant in all our analyses and equations in this work which is also true in our previous works in [1, 2, 3, 4, 5].

In Sec. II we present the metric and fluid quantities perturbed to the third order which will be required in our calculation. We present fluid quantities for most general fluids with pressures and stresses which will turn out to be important even in the zero-pressure situation in our main analysis. As in the single component case in [4], under our proper choice of variables and gauges we do not need third-order perturbations of the connection or curvature tensor. In Sec. III we summarize the relativistic equations to the second order, and their effective correspondence with the Newtonian ones even in the multi-component situation. In Sec. IV we derive the general relativistic third-order terms and present equations in the context of Newtonian gravity with pure general relativistic corrections. We compare our equations in the multi-component case with the previously derived ones in a single component. Section V is a discussion. We often set $c \equiv 1$, but recover c in the Newtonian context presentation.

II.   THIRD-ORDER PERTURBATIONS

A.   The covariant and ADM equations

In the following we summarize the basic sets of covariant equations and ADM equations we need in our analysis. These equations, except for the covariant equations of individual component, are presented Sec. II of [1]; notations can be found in that work. For original studies of the covariant and the ADM equations, see [12], and [13], respectively. Although, we will use the ADM equations in our calculation, the covariant equations show another aspects of the same fully nonlinear system of Einstein's equation.

The energy-momentum tenor of a fluid can be decomposed into fluid quantities as

$$\tilde{T}_{ab} = \tilde{\mu}\tilde{u}_a\tilde{u}_b + \tilde{p}\left(\tilde{u}_a\tilde{u}_b + \tilde{g}_{ab}\right) + \tilde{q}_a\tilde{u}_b + \tilde{q}_b\tilde{u}_a + \tilde{\pi}_{ab}. \tag{1}$$

Without losing generality, we take the energy frame, thus set $\tilde{q}_a \equiv 0$. This decomposition is valid even in the multiple component fluids; in such a case the above fluid quantities can be regarded as collective fluid quantities. In the multi-component case we introduce the energy-momentum tensor and fluid quantities of individual component as

$$\tilde{T}_{ab} \equiv \sum_j \tilde{T}_{(j)ab},$$

$$\tilde{T}_{(i)ab} = \tilde{\mu}_{(i)}\tilde{u}_{(i)a}\tilde{u}_{(i)b} + \tilde{p}_{(i)}\left(\tilde{u}_{(i)a}\tilde{u}_{(i)b} + \tilde{g}_{ab}\right) + \tilde{\pi}_{(i)ab}, \tag{2}$$

where, without losing generality, we also took the energy-frame condition for each component, thus set $\tilde{q}_{(i)a} \equiv 0$. For interactions among components we introduce

$$\tilde{T}_{(i)a;b}^{\ \ \ b} \equiv \tilde{I}_{(i)a}, \quad \sum_j \tilde{I}_{(j)a} = 0. \tag{3}$$

In a single component situation, taking the energy-frame (thus, setting $\tilde{q}_a \equiv 0$), the energy conservation equation, the momentum conservation equation, and the Raychaudhury equation are

$$\dot{\tilde{\mu}} + \left(\tilde{\mu} + \tilde{p}\right)\tilde{\theta} + \tilde{\pi}^{ab}\tilde{\sigma}_{ab} = 0, \tag{4}$$

$$\left(\tilde{\mu} + \tilde{p}\right)\tilde{a}_a + \tilde{h}_a^b\left(\tilde{p}_{,b} + \tilde{\pi}_{b;c}^c\right) = 0, \tag{5}$$

$$\dot{\tilde{\theta}} + \frac{1}{3}\tilde{\theta}^2 - \tilde{a}^a_{\ ;a} + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - \tilde{\omega}^{ab}\tilde{\omega}_{ab} + 4\pi G\left(\tilde{\mu} + 3\tilde{p}\right) - \Lambda = 0. \tag{6}$$

In the multi-component case, taking the energy-frame for individual component (thus, setting $\tilde{q}_{(i)a} \equiv 0$) we have

$$\dot{\tilde{\mu}}_{(i)} + \left(\tilde{\mu}_{(i)} + \tilde{p}_{(i)}\right)\tilde{\theta}_{(i)} + \tilde{\pi}_{(i)}^{ab}\tilde{\sigma}_{(i)ab} = -\tilde{u}_{(i)}^a\tilde{I}_{(i)a}, \tag{7}$$

$$\left(\tilde{\mu}_{(i)} + \tilde{p}_{(i)}\right)\tilde{a}_{(i)a} + \tilde{h}_{(i)a}^{\ \ b}\left(\tilde{p}_{(i),b} + \tilde{\pi}_{(i)b;c}^c\right) = \tilde{h}_{(i)a}^{\ \ b}\tilde{I}_{(i)b}, \tag{8}$$

$$\dot{\tilde{\theta}}_{(i)} + \frac{1}{3}\tilde{\theta}_{(i)}^2 - \tilde{a}_{(i);a}^a + \tilde{\sigma}_{(i)}^{ab}\tilde{\sigma}_{(i)ab} - \tilde{\omega}_{(i)}^{ab}\tilde{\omega}_{(i)ab} - 4\pi G\left(\tilde{\mu} - 3\tilde{p} - 2\tilde{T}_{ab}\tilde{u}_{(i)}^a\tilde{u}_{(i)}^b\right) - \Lambda = 0. \tag{9}$$

In order to handle cosmological perturbations to the third order, we also need the momentum constraint equation for a collective fluid

$$\tilde{h}_{ab}\left(\tilde{\omega}^{bc}_{\ \ ;c} - \tilde{\sigma}^{bc}_{\ \ ;c} + \frac{2}{3}\tilde{\theta}^{,b}\right) + \left(\tilde{\omega}_{ab} + \tilde{\sigma}_{ab}\right)\tilde{a}^b = 0. \tag{10}$$

In the ADM formulation, the energy conservation, momentum conservation, and trace of ADM propagation equations are

$$E_{,0}N^{-1} - E_{,\alpha}N^\alpha N^{-1} - K\left(E + \frac{1}{3}S\right) - \bar{S}^{\alpha\beta}\bar{K}_{\alpha\beta} + N^{-2}\left(N^2J^\alpha\right)_{:\alpha} = 0, \tag{11}$$

$$J_{\alpha,0}N^{-1} - J_{\alpha:\beta}N^\beta N^{-1} - J_\beta N^\beta_{\ :\alpha}N^{-1} - KJ_\alpha + EN_{,\alpha}N^{-1} + S^\beta_{\alpha:\beta} + S^\beta_\alpha N_{,\beta}N^{-1} = 0, \tag{12}$$

$$K_{,0}N^{-1} - K_{,\alpha}N^\alpha N^{-1} + N^{:\alpha}_{\ \ \alpha}N^{-1} - \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - \frac{1}{3}K^2 - 4\pi G\left(E + S\right) + \Lambda = 0. \tag{13}$$

In the multi-component case we have

$$E_{(i),0}N^{-1} - E_{(i),\alpha}N^\alpha N^{-1} - K\left(E_{(i)} + \frac{1}{3}S_{(i)}\right) - \bar{S}_{(i)}^{\alpha\beta}\bar{K}_{\alpha\beta} + N^{-2}\left(N^2J_{(i)}^\alpha\right)_{:\alpha} = -\frac{1}{N}\left(\tilde{I}_{(i)0} - \tilde{I}_{(i)\alpha}N^\alpha\right), \tag{14}$$

$$J_{(i)\alpha,0}N^{-1} - J_{(i)\alpha:\beta}N^\beta N^{-1} - J_{(i)\beta}N^\beta_{\ :\alpha}N^{-1} - KJ_{(i)\alpha} + E_{(i)}N_{,\alpha}N^{-1} + S_{(i)\alpha:\beta}^{\ \ \ \beta} + S_{(i)\alpha}^{\ \ \beta}N_{,\beta}N^{-1} = \tilde{I}_{(i)\alpha}. \tag{15}$$

The momentum constraint equation is

$$\bar{K}^\beta_{\alpha:\beta} - \frac{2}{3}K_{,\alpha} = 8\pi GJ_\alpha. \tag{16}$$

The above sets of equations are only parts of the covariant and the ADM equations; for complete sets, see [1, 12, 13, 14, 15]. We will show that, by taking proper choice of gauges, the scalar-type perturbations to the third order can be derived from either of the above sets of equations. We will present the derivation based on the ADM equations, because the covariant formalism often requires lengthier calculation in our particular case; of course, the covariant equations also give the same result. As we will show, however, we use both formulations simultaneously depending on the convenience.

B. Metric

Our metric convention is the same as in [1]

$$\tilde{g}_{00} = -a^2 \left(1 + 2A\right), \quad \tilde{g}_{0\alpha} = -a^2 B_\alpha, \quad \tilde{g}_{\alpha\beta} = a^2 \left(g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta}\right), \tag{17}$$

where tensor indices of B_α and $C_{\alpha\beta}$ are based on $g^{(3)}_{\alpha\beta}$. To the third order, the inverse metric becomes

$$\tilde{g}^{00} = -\frac{1}{a^2}\left(1 - 2A + 4A^2 - B^\alpha B_\alpha - 8A^3 + 4AB^\alpha B_\alpha + 2B^\alpha B^\beta C_{\alpha\beta}\right),$$

$$\tilde{g}^{0\alpha} = -\frac{1}{a^2}\left[B^\alpha - 2AB^\alpha - 2B^\beta C^\alpha_\beta + B^\alpha\left(4A^2 - B^\beta B_\beta\right) + 4C^\alpha_\beta\left(AB^\beta + B^\gamma C^\beta_\gamma\right)\right],$$

$$\tilde{g}^{\alpha\beta} = \frac{1}{a^2}\left(g^{(3)\alpha\beta} - 2C^{\alpha\beta} - B^\alpha B^\beta + 4C^\alpha_\gamma C^{\beta\gamma} + 2AB^\alpha B^\beta + 2B^\alpha B^\gamma C^\beta_\gamma + 2B^\beta B^\gamma C^\alpha_\gamma - 8C^\alpha_\gamma C^\beta_\delta C^{\gamma\delta}\right). \tag{18}$$

In order to derive perturbation equations to the third order, we need the connection only to the second-order. These are presented in Eq. (52) of [1].

The ADM metric variables follow from Eq. (2) of [1] as

$$N = a\left(1 + A - \frac{1}{2}A^2 + \frac{1}{2}B^\alpha B_\alpha + \frac{1}{2}A^3 - \frac{1}{2}AB^\alpha B_\alpha - B^\alpha B^\beta C_{\alpha\beta}\right),$$

$$N_\alpha = -a^2 B_\alpha, \quad N^\alpha = -B^\alpha + 2B^\beta C^\alpha_\beta - 4B^\beta C^\alpha_\gamma C^\gamma_\beta,$$

$$h_{\alpha\beta} = a^2\left(g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta}\right), \quad h^{\alpha\beta} = \frac{1}{a^2}\left(g^{(3)\alpha\beta} - 2C^{\alpha\beta} + 4C^\alpha_\gamma C^{\beta\gamma} - 8C^\alpha_\gamma C^\beta_\delta C^{\gamma\delta}\right), \tag{19}$$

where tensor index of N_α is based on $h_{\alpha\beta}$ as the metric; $h^{\alpha\beta}$ is an inverse metric of $h_{\alpha\beta}$.

C. Fluid quantities

We introduce perturbations of fluid quantities as

$$\tilde{\mu} = \mu + \delta\mu, \quad \tilde{p} = p + \delta p, \quad \tilde{u}_\alpha \equiv av_\alpha, \quad \tilde{\pi}_{\alpha\beta} \equiv a^2\Pi_{\alpha\beta}, \tag{20}$$

where tensor indices of v_α and $\Pi_{\alpha\beta}$ are based on $g^{(3)}_{\alpha\beta}$. The above fluid quantities can be regarded as collective fluid quantities in the case of multi-component fluids, see below Eq. (26). Although we will consider zero-pressure fluids, it is important to keep the perturbed pressure (δp) and anisotropic stress ($\Pi_{\alpha\beta}$) because the collective pressure and anisotropic stress do not vanish to nonlinear order in the multi-component fluids even in the zero-pressure case, see Eq. (44). In any case, for later convenience, in this section we will present fluid quantities for most general fluids.

Components of the four-vector $\tilde{u}_a$ are

$$\tilde{u}_\alpha \equiv av_\alpha,$$

$$\tilde{u}_0 = -a\left[1 + A - \frac{1}{2}A^2 + \frac{1}{2}\left(v^\alpha + B^\alpha\right)\left(v_\alpha + B_\alpha\right) + \frac{1}{2}A^3 + \frac{1}{2}A\left(v^\alpha v_\alpha - B^\alpha B_\alpha\right) - C_{\alpha\beta}\left(v^\alpha + B^\alpha\right)\left(v^\beta + B^\beta\right)\right],$$

$$\tilde{u}^\alpha = \frac{1}{a}\left[v^\alpha + B^\alpha - AB^\alpha - 2C^{\alpha\beta}\left(v_\beta + B_\beta\right) + \frac{3}{2}A^2 B^\alpha + 2AB^\beta C^\alpha_\beta + \frac{1}{2}B^\alpha\left(v^\beta v_\beta - B^\beta B_\beta\right) + 4C^\alpha_\beta C^\beta_\gamma\left(v^\gamma + B^\gamma\right)\right],$$

$$\tilde{u}^0 = \frac{1}{a}\left[1 - A + \frac{3}{2}A^2 + \frac{1}{2}\left(v^\alpha v_\alpha - B^\alpha B_\alpha\right) - \frac{5}{2}A^3 - \frac{1}{2}A\left(v^\alpha v_\alpha - 3B^\alpha B_\alpha\right) - C_{\alpha\beta}\left(v^\alpha v^\beta - B^\alpha B^\beta\right)\right]. \tag{21}$$

In [1], instead of v_α, we used V_α defined as

$$\tilde{u}^\alpha \equiv \frac{1}{a} V^\alpha. \tag{22}$$

Thus, we have

$$v_\alpha = V_\alpha - B_\alpha + AB_\alpha + 2C_{\alpha\beta}V^\beta - \frac{3}{2}A^2 B_\alpha - B_\alpha V_\beta \left(\frac{1}{2}V^\beta - B^\beta \right). \tag{23}$$

Components of $\tilde{\pi}_{ab}$ are

$$\tilde{\pi}_{\alpha\beta} \equiv a^2 \Pi_{\alpha\beta}, \quad \tilde{\pi}_{0\alpha} = -a^2 \Pi_{\alpha\beta} \left[v^\beta + B^\beta + Av^\beta - 2C^{\beta\gamma}(v_\gamma + B_\gamma) \right], \quad \tilde{\pi}_{00} = a^2 \Pi_{\alpha\beta} (v^\alpha + B^\alpha)(v^\beta + B^\beta). \tag{24}$$

From $\tilde{\pi}_c^c \equiv 0$ we have

$$\Pi_\alpha^\alpha - 2C^{\alpha\beta}\Pi_{\alpha\beta} + \left(4C_\gamma^\alpha C^{\beta\gamma} - v^\alpha v^\beta \right) \Pi_{\alpha\beta} = 0. \tag{25}$$

To the third order, the energy-momentum tensor becomes

$$\tilde{T}_0^0 = -\mu - \delta\mu - (\mu + p)v^\alpha (v_\alpha + B_\alpha) + (\mu + p) \left[AB_\alpha + 2C_{\alpha\beta}(v^\beta + B^\beta) \right] v^\alpha$$
$$\quad - (\delta\mu + \delta p)v^\alpha (v_\alpha + B_\alpha) - \Pi_{\alpha\beta}v^\alpha (v^\beta + B^\beta),$$

$$\tilde{T}_\alpha^0 = (\mu + p)v_\alpha - (\mu + p)Av_\alpha + (\delta\mu + \delta p)v_\alpha + \Pi_{\alpha\beta}v^\beta + \frac{1}{2}(\mu + p)\left(3A^2 + v^\beta v_\beta - B^\beta B_\beta \right)v_\alpha$$
$$\quad - (\delta\mu + \delta p)Av_\alpha - \Pi_{\alpha\beta}\left(Av^\beta + 2C^{\beta\gamma}v_\gamma \right),$$

$$\tilde{T}_\beta^\alpha = p\delta_\beta^\alpha + \delta p\delta_\beta^\alpha + \Pi_\beta^\alpha + (\mu + p)(v^\alpha + B^\alpha)v_\beta - 2C^{\alpha\gamma}\Pi_{\beta\gamma} - (\mu + p)\left[AB^\alpha + 2C^{\alpha\gamma}(v_\gamma + B_\gamma) \right]v_\beta$$
$$\quad + (\delta\mu + \delta p)(v^\alpha + B^\alpha)v_\beta + \Pi_{\beta\gamma}\left(B^\alpha v^\gamma + 4C_\delta^\alpha C^{\gamma\delta} \right). \tag{26}$$

Above relations of four-vectors, energy-momentum tensor, and fluid quantities are derived for a single component fluid. However, these are also valid as the collective component in the case of multi-component fluids. We can easily see that, by replacing these quantities with the ones of individual component, the same relations are valid for individual component as well. That is, for an individual component, say i-th component, by replacing

$$\tilde{T}_{ab}, \quad \tilde{\mu}, \quad \tilde{p}, \quad \tilde{u}_a, \quad \tilde{\pi}_{ab}, \quad \mu, \quad p, \quad \delta\mu, \quad \delta p, \quad V_\alpha, \quad v_\alpha, \quad \Pi_{\alpha\beta}, \tag{27}$$

with

$$\tilde{T}_{(i)ab}, \quad \tilde{\mu}_{(i)}, \quad \tilde{p}_{(i)}, \quad \tilde{u}_{(i)a}, \quad \tilde{\pi}_{(i)ab}, \quad \mu_{(i)}, \quad p_{(i)}, \quad \delta\mu_{(i)}, \quad \delta p_{(i)}, \quad V_{(i)\alpha}, \quad v_{(i)\alpha}, \quad \Pi_{(i)\alpha\beta}, \tag{28}$$

Eqs. (20)-(26) are valid for the i-th component.

From Eqs. (1),(26) we can express the collective fluid quantities in terms of the individual one. To the background order we have

$$\mu = \sum_j \mu_{(j)}, \quad p = \sum_j p_{(j)}. \tag{29}$$

To the third order in perturbations, we can show

$$
\begin{aligned}
&\delta\mu + \left[(\mu+p)\,v^\alpha + (\delta\mu+\delta p)\,v^\alpha + \Pi^\alpha_\beta v^\beta\right]v_\alpha - 2\,(\mu+p)\,C_{\alpha\beta}v^\alpha v^\beta \\
&\qquad = \sum_j \left\{\delta\mu_{(j)} + \left[\left(\mu_{(j)}+p_{(j)}\right)v^\alpha_{(j)} + \left(\delta\mu_{(j)}+\delta p_{(j)}\right)v^\alpha_{(j)} + \Pi_{(j)\beta}^{\ \ \alpha}v^\beta_{(j)}\right]v_{(j)\alpha} - 2\left(\mu_{(j)}+p_{(j)}\right)C_{\alpha\beta}v^\alpha_{(j)}v^\beta_{(j)}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\delta p + \frac{1}{3}\left[(\mu+p)\,v^\alpha + (\delta\mu+\delta p)\,v^\alpha + \Pi^\alpha_\beta v^\beta\right]v_\alpha - \frac{2}{3}\,(\mu+p)\,C_{\alpha\beta}v^\alpha v^\beta \\
&\qquad = \sum_j \left\{\delta p_{(j)} + \frac{1}{3}\left[\left(\mu_{(j)}+p_{(j)}\right)v^\alpha_{(j)} + \left(\delta\mu_{(j)}+\delta p_{(j)}\right)v^\alpha_{(j)} + \Pi_{(j)\beta}^{\ \ \alpha}v^\beta_{(j)}\right]v_{(j)\alpha} - \frac{2}{3}\left(\mu_{(j)}+p_{(j)}\right)C_{\alpha\beta}v^\alpha_{(j)}v^\beta_{(j)}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&(\mu+p)\,v_\alpha + \left[\delta\mu + \delta p + \frac{4}{3}\,(\mu+p)\,v^\beta v_\beta\right]v_\alpha + \left[\Pi^\beta_\alpha + (\mu+p)\left(v_\alpha v^\beta - \frac{1}{3}\delta^\beta_\alpha v^\gamma v_\gamma\right)\right]v_\beta - \frac{3}{2}\,(\mu+p)\,v_\alpha v^\beta v_\beta \\
&\quad -2\Pi_{\alpha\beta}C^{\beta\gamma}v_\gamma = \sum_j \left\{ \left(\mu_{(j)}+p_{(j)}\right)v_{(j)\alpha} + \left[\delta\mu_{(j)} + \delta p_{(j)} + \frac{4}{3}\left(\mu_{(j)}+p_{(j)}\right)v^\beta_{(j)}v_{(j)\beta}\right]v_{(j)\alpha}\right. \\
&\qquad\left. + \left[\Pi_{(j)\alpha}^{\ \ \beta} + \left(\mu_{(j)}+p_{(j)}\right)\left(v_{(j)\alpha}v^\beta_{(j)} - \frac{1}{3}\delta^\beta_\alpha v^\gamma_{(j)}v_{(j)\gamma}\right)\right]v_{(j)\beta} - \frac{3}{2}\left(\mu_{(j)}+p_{(j)}\right)v_{(j)\alpha}v^\beta_{(j)}v_{(j)\beta} - 2\Pi_{(j)\alpha\beta}C^{\beta\gamma}v_{(j)\gamma}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\Pi^\alpha_\beta + \left[(\mu+p)\,v^\alpha + (\delta\mu+\delta p)\,v^\alpha + \Pi^\alpha_\gamma v^\gamma\right]v_\beta - \frac{2}{3}\,(\mu+p)\,C^\alpha_\beta v^\gamma v_\gamma - \Pi^{\alpha\gamma}v_\beta v_\gamma \\
&\quad -\frac{1}{3}\delta^\alpha_\beta\left\{\left[(\mu+p)\,v^\gamma + (\delta\mu+\delta p)\,v^\gamma + \Pi^\gamma_\delta v^\delta\right]v_\gamma - 2\,(\mu+p)\,C_{\gamma\delta}v^\gamma v^\delta\right\} \\
&\qquad = \sum_j \left\{\Pi_{(j)\beta}^{\ \ \alpha} + \left[\left(\mu_{(j)}+p_{(j)}\right)v^\alpha_{(j)} + \left(\delta\mu_{(j)}+\delta p_{(j)}\right)v^\alpha_{(j)} + \Pi_{(j)\gamma}^{\ \ \alpha}v^\gamma_{(j)}\right]v_{(j)\beta}\right. \\
&\qquad\quad -\frac{2}{3}\left(\mu_{(j)}+p_{(j)}\right)C^\alpha_\beta v^\gamma_{(j)}v_{(j)\gamma} - \Pi_{(j)}^{\alpha\gamma}v_{(j)\beta}v_{(j)\gamma} \\
&\qquad\quad \left.-\frac{1}{3}\delta^\alpha_\beta\left\{\left[\left(\mu_{(j)}+p_{(j)}\right)v^\gamma_{(j)} + \left(\delta\mu_{(j)}+\delta p_{(j)}\right)v^\gamma_{(j)} + \Pi_{(j)\delta}^{\ \ \gamma}v^\delta_{(j)}\right]v_{(j)\gamma} - 2\left(\mu_{(j)}+p_{(j)}\right)C_{\gamma\delta}v^\gamma_{(j)}v^\delta_{(j)}\right\}\right\}.
\end{aligned}
\tag{30}
$$

Thus, we have

$$
\begin{aligned}
\delta\mu = \sum_j &\left\{\delta\mu_{(j)} + \left(\mu_{(j)}+p_{(j)}\right)v^\alpha_{(j)}\left(v_{(j)\alpha} - v_\alpha\right)\right. \\
&\left. + \left[\left(\delta\mu_{(j)}+\delta p_{(j)}\right)v^\alpha_{(j)} + \Pi_{(j)\beta}^{\ \ \alpha}v^\beta_{(j)} - 2\left(\mu_{(j)}+p_{(j)}\right)C^\alpha_\beta v^\beta_{(j)}\right]\left(v_{(j)\alpha} - v_\alpha\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\delta p = \sum_j &\left\{\delta p_{(j)} + \frac{1}{3}\left(\mu_{(j)}+p_{(j)}\right)v^\alpha_{(j)}\left(v_{(j)\alpha} - v_\alpha\right)\right. \\
&\left. + \frac{1}{3}\left[\left(\delta\mu_{(j)}+\delta p_{(j)}\right)v^\alpha_{(j)} + \Pi_{(j)\beta}^{\ \ \alpha}v^\beta_{(j)} - 2\left(\mu_{(j)}+p_{(j)}\right)C^\alpha_\beta v^\beta_{(j)}\right]\left(v_{(j)\alpha} - v_\alpha\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
(\mu+p)\,v_\alpha = \sum_j &\left\{\left(\mu_{(j)}+p_{(j)}\right)v_{(j)\alpha} + \left(\delta\mu_{(j)}+\delta p_{(j)}\right)\left(v_{(j)\alpha} - v_\alpha\right) + \Pi_{(j)\alpha}^{\ \ \beta}\left(v_{(j)\beta} - v_\beta\right)\right. \\
&\left. + \left(\mu_{(j)}+p_{(j)}\right)v^\beta_{(j)}v_{(j)\beta}\left(v_{(j)\alpha} - v_\alpha\right) - \left[\frac{1}{2}\left(\mu_{(j)}+p_{(j)}\right)v_{(j)\alpha}\left(v^\beta_{(j)} + 3v^\beta\right) + 2\Pi_{(j)\alpha\beta}C^{\beta\gamma}\right]\left(v_{(j)\beta} - v_\beta\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\Pi^\alpha_\beta = \sum_j &\left\{\Pi_{(j)\beta}^{\ \ \alpha} + \left(\mu_{(j)}+p_{(j)}\right)\left[v^\alpha_{(j)}\left(v_{(j)\beta} - v_\beta\right) - \frac{1}{3}\delta^\alpha_\beta v^\gamma_{(j)}\left(v_{(j)\gamma} - v_\gamma\right)\right]\right. \\
&+ \left[\left(\delta\mu_{(j)}+\delta p_{(j)}\right)v^\alpha_{(j)} - \Pi_{(j)\gamma}^{\ \ \alpha}v^\gamma\right]\left(v_{(j)\beta} - v_\beta\right) - \frac{2}{3}\left(\mu_{(j)}+p_{(j)}\right)C^\alpha_\beta v^\gamma_{(j)}\left(v_{(j)\gamma} - v_\gamma\right) \\
&\left.-\frac{1}{3}\delta^\alpha_\beta\left[\left(\delta\mu_{(j)}+\delta p_{(j)}\right)v^\gamma_{(j)} + \Pi_{(j)\delta}^{\ \ \gamma}v^\delta_{(j)} - 2\left(\mu_{(j)}+p_{(j)}\right)C^\gamma_\delta v^\delta_{(j)}\right]\left(v_{(j)\gamma} - v_\gamma\right)\right\}.
\end{aligned}
\tag{31}
$$

The ADM fluid quantities are introduced as

$$E \equiv \tilde{n}_a \tilde{n}_b \tilde{T}^{ab}, \quad J_\alpha \equiv -\tilde{n}_b \tilde{T}^b_\alpha, \quad S_{\alpha\beta} \equiv \tilde{T}_{\alpha\beta}, \quad S \equiv h^{\alpha\beta} S_{\alpha\beta}, \quad \bar{S}_{\alpha\beta} \equiv S_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} S, \tag{32}$$

where tensor indices of J_α and $S_{\alpha\beta}$ are based on $h_{\alpha\beta}$ as the metric. The four-vector $\tilde{n}_a$ is a normal frame four-vector with $\tilde{n}_\alpha \equiv 0$. Thus by setting $v_\alpha \equiv 0$ we have $\tilde{u}_a = \tilde{n}_a$, and Eq. (21) gives

$$\tilde{n}_\alpha \equiv 0,$$
$$\tilde{n}_0 = -a \left(1 + A - \frac{1}{2} A^2 + \frac{1}{2} B^\alpha B_\alpha + \frac{1}{2} A^3 - \frac{1}{2} AB^\alpha B_\alpha - C_{\alpha\beta} B^\alpha B^\beta \right),$$
$$\tilde{n}^\alpha = \frac{1}{a} \left(B^\alpha - AB^\alpha - 2C^{\alpha\beta} B_\beta + \frac{3}{2} A^2 B^\alpha + 2AB^\beta C^\alpha_\beta - \frac{1}{2} B^\alpha B^\beta B_\beta + 4C^\alpha_\beta C^\beta_\gamma B^\gamma \right),$$
$$\tilde{n}^0 = \frac{1}{a} \left(1 - A + \frac{3}{2} A^2 - \frac{1}{2} B^\alpha B_\alpha - \frac{5}{2} A^3 + \frac{3}{2} AB^\alpha B_\alpha + C_{\alpha\beta} B^\alpha B^\beta \right). \tag{33}$$

Using Eqs. (26),(33), Eq. (32) gives

$$E = \mu + \delta\mu + (\mu + p) v^\alpha v_\alpha + (\delta\mu + \delta p) v^\alpha v_\alpha - 2 (\mu + p) C_{\alpha\beta} v^\alpha v^\beta + \Pi_{\alpha\beta} v^\alpha v^\beta,$$
$$J_\alpha = a \left[(\mu + p) v_\alpha + (\delta\mu + \delta p) v_\alpha + \Pi_{\alpha\beta} v^\beta + \frac{1}{2} (\mu + p) v_\alpha v^\beta v_\beta - 2\Pi_{\alpha\beta} C^\beta_\gamma v^\gamma \right],$$
$$S_{\alpha\beta} = a^2 \left[pg^{(3)}_{\alpha\beta} + \delta p g^{(3)}_{\alpha\beta} + \Pi_{\alpha\beta} + 2pC_{\alpha\beta} + (\mu + p) v_\alpha v_\beta + 2\delta p C_{\alpha\beta} + (\delta\mu + \delta p) v_\alpha v_\beta \right],$$
$$S = 3p + 3\delta p + (\mu + p) v^\alpha v_\alpha + (\delta\mu + \delta p) v^\alpha v_\alpha + v^\alpha v^\beta \left[\Pi_{\alpha\beta} - 2 (\mu + p) C_{\alpha\beta} \right],$$
$$\bar{S}_{\alpha\beta} = a^2 \left\{ \Pi_{\alpha\beta} + (\mu + p) \left(v_\alpha v_\beta - \frac{1}{3} g^{(3)}_{\alpha\beta} v^\gamma v_\gamma \right) + (\delta\mu + \delta p) v_\alpha v_\beta - \frac{2}{3} (\mu + p) C_{\alpha\beta} v^\gamma v_\gamma \right.$$
$$\left. - \frac{1}{3} g^{(3)}_{\alpha\beta} \left[(\delta\mu + \delta p) v^\gamma v_\gamma + \Pi_{\gamma\delta} v^\gamma v^\delta - 2 (\mu + p) C_{\gamma\delta} v^\gamma v^\delta \right] \right\}. \tag{34}$$

The individual ADM fluid quantities can be found by replacing

$$E, \quad J_\alpha, \quad S_{\alpha\beta}, \quad S, \quad \bar{S}_{\alpha\beta}, \tag{35}$$

with

$$E_{(i)}, \quad J_{(i)\alpha}, \quad S_{(i)\alpha\beta}, \quad S_{(i)}, \quad \bar{S}_{(i)\alpha\beta}, \tag{36}$$

and similarly for the fluid quantities and the energy-momentum tensor as in Eqs. (27),(28). From Eq. (2) we have

$$E = \sum_j E_{(j)}, \quad J_\alpha = \sum_j J_{(j)\alpha}, \quad S_{\alpha\beta} = \sum_j S_{(j)\alpha\beta}, \quad S = \sum_j S_{(j)}, \quad \bar{S}_{\alpha\beta} = \sum_j \bar{S}_{(j)\alpha\beta}. \tag{37}$$

D. Decomposition

We decompose the metric into three perturbation types [16]

$$A \equiv \alpha, \quad B_\alpha \equiv \beta_{,\alpha} + B^{(v)}_\alpha, \quad C_{\alpha\beta} \equiv \varphi g^{(3)}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(v)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta}, \tag{38}$$

where superscripts (v) and (t) indicate the transverse vector-type, and transverse-tracefree tensor-type perturbations, respectively. Only to the linear-order perturbations in the homogeneous-isotropic background, these three types of perturbations decouple and evolve independently. We introduce

$$\chi \equiv a \left(\beta + c^{-1} a\dot{\gamma} \right), \quad \Psi^{(v)}_\alpha \equiv B^{(v)}_\alpha + c^{-1} a\dot{C}^{(v)}_\alpha, \tag{39}$$

which are spatially gauge-invariant combinations to the linear order [14]. We set

$$K \equiv -3H + \kappa. \tag{40}$$

By using κ we can avoid third-order expansion of the trace of extrinsic curvature K. Identifying κ with Newtonian velocity variable later will be an important step in our analysis, see Eqs. (63),(101).

For the fluid quantities we decompose

$$v_\alpha \equiv -v_{,\alpha} + v_\alpha^{(v)}, \quad \Pi_{\alpha\beta} \equiv \frac{1}{a^2}\left(\Pi_{,\alpha|\beta} - \frac{1}{3}g_{\alpha\beta}^{(3)}\Delta\Pi\right) + \frac{1}{a}\Pi_{(\alpha|\beta)}^{(v)} + \Pi_{\alpha\beta}^{(t)},$$

$$v_{(i)\alpha} \equiv -v_{(i),\alpha} + v_{(i)\alpha}^{(v)}, \quad \Pi_{(i)\alpha\beta} \equiv \frac{1}{a^2}\left(\Pi_{(i),\alpha|\beta} - \frac{1}{3}g_{\alpha\beta}^{(3)}\Delta\Pi_{(i)}\right) + \frac{1}{a}\Pi_{(i)(\alpha|\beta)}^{(v)} + \Pi_{(i)\alpha\beta}^{(t)}, \quad \delta I_{(i)\alpha} \equiv \delta I_{(i),\alpha} + \delta I_{(i)\alpha}^{(v)} \tag{41}$$

The perturbed fluid velocity variables v and $v_\alpha^{(v)}$ subtly differ from the ones introduced in [1], see [5].

E.   Zero-pressure irrotational fluids in the comoving gauge

The zero-pressure condition sets

$$p_{(i)} \equiv 0, \quad \delta p_{(i)} \equiv 0 \equiv \Pi_{(i)\alpha\beta}. \tag{42}$$

The comoving gauge condition ($v \equiv 0$) and irrotational condition ($v_\alpha^{(v)} = 0$) give

$$v_\alpha \equiv 0. \tag{43}$$

Under these conditions, Eq. (31) becomes

$$\delta\mu = \sum_j \left(\delta\mu_{(j)} + \mu_{(j)}v_{(j)}^\alpha v_{(j)\alpha} + \delta\mu_{(j)}v_{(j)}^\alpha v_{(j)\alpha} - 2\mu_{(j)}C_{\alpha\beta}v_{(j)}^\alpha v_{(j)}^\beta\right),$$

$$\delta p = \frac{1}{3}\sum_j \left(\mu_{(j)}v_{(j)}^\alpha v_{(j)\alpha} + \delta\mu_{(j)}v_{(j)}^\alpha v_{(j)\alpha} - 2\mu_{(j)}C_{\alpha\beta}v_{(j)}^\alpha v_{(j)}^\beta\right),$$

$$0 = \sum_j \left(\mu_{(j)}v_{(j)\alpha} + \delta\mu_{(j)}v_{(j)\alpha} + \frac{1}{2}\mu_{(j)}v_{(j)\alpha}v_{(j)\beta}^\beta v_{(j)\beta}\right),$$

$$\Pi_\beta^\alpha = \sum_j \left[\mu_{(j)}\left(v_{(j)}^\alpha v_{(j)\beta} - \frac{1}{3}\delta_\beta^\alpha v_{(j)}^\gamma v_{(j)\gamma}\right) + \delta\mu_{(j)}\left(v_{(j)}^\alpha v_{(j)\beta} - \frac{1}{3}\delta_\beta^\alpha v_{(j)}^\gamma v_{(j)\gamma}\right)\right.$$
$$\left. - \frac{2}{3}\mu_{(j)}\left(C_\beta^\alpha v_{(j)}^\gamma v_{(j)\gamma} - \delta_\beta^\alpha C_\delta^\gamma v_{(j)}^\delta v_{(j)\gamma}\right)\right]. \tag{44}$$

Notice that for the nonlinear perturbations, the collective pressure δp and anisotropic stress $\Pi_{\alpha\beta}$ no longer vanish even for zero-pressure fluids; also, the collective energy density $\delta\mu$ is no longer a simple sum of individual component.

For the ADM fluid quantities, Eq. (34) becomes

$$E = \mu + \delta\mu, \quad J_\alpha = 0, \quad S_{\alpha\beta} = a^2\left(\delta p g_{\alpha\beta}^{(3)} + \Pi_{\alpha\beta} + 2\delta p C_{\alpha\beta}\right), \quad S = 3\delta p, \quad \bar{S}_{\alpha\beta} = a^2\Pi_{\alpha\beta}. \tag{45}$$

For the individual component, Eq. (34) gives

$$E_{(i)} = \mu_{(i)} + \delta\mu_{(i)} + \mu_{(i)}v_{(i)}^\alpha v_{(i)\alpha} + \delta\mu_{(i)}v_{(i)}^\alpha v_{(i)\alpha} - 2\mu_{(i)}C_{\alpha\beta}v_{(i)}^\alpha v_{(i)}^\beta,$$

$$J_{(i)\alpha} = a\left(\mu_{(i)}v_{(i)\alpha} + \delta\mu_{(i)}v_{(i)\alpha} + \frac{1}{2}\mu_{(i)}v_{(i)\alpha}v_{(i)}^\beta v_{(i)\beta}\right),$$

$$S_{(i)\alpha\beta} = a^2\left(\mu_{(i)}v_{(i)\alpha}v_{(i)\beta} + \delta\mu_{(i)}v_{(i)\alpha}v_{(i)\beta}\right),$$

$$S_{(i)} = \mu_{(i)}v_{(i)}^\alpha v_{(i)\alpha} + \delta\mu_{(i)}v_{(i)}^\alpha v_{(i)\alpha} - 2\mu_{(i)}C_{\alpha\beta}v_{(i)}^\alpha v_{(i)}^\beta,$$

$$\bar{S}_{(i)\alpha\beta} = a^2\left[\mu_{(i)}\left(v_{(i)\alpha}v_{(i)\beta} - \frac{1}{3}g_{\alpha\beta}^{(3)}v_{(i)}^\gamma v_{(i)\gamma}\right) + \delta\mu_{(i)}v_{(i)\alpha}v_{(i)\beta} - \frac{2}{3}\mu_{(i)}C_{\alpha\beta}v_{(i)}^\gamma v_{(i)\gamma}\right.$$
$$\left. - \frac{1}{3}g_{\alpha\beta}^{(3)}\left(\delta\mu_{(i)}v_{(i)}^\gamma v_{(i)\gamma} - 2\mu_{(i)}C_{\gamma\delta}v_{(i)}^\gamma v_{(i)}^\delta\right)\right]. \tag{46}$$

III.   SECOND-ORDER EQUATIONS: SUMMARY

In this section we summarize our previous work of second-order perturbations in zero-pressure, irrotational, but multi-component fluids [5]. The results show that, to the second order, effectively the relativistic equations coincide with the Newtonian ones even in the multi-component situation. This provides a reason to go to the third order in relativistic perturbation in order to find out pure general relativistic deviations from Newton's theory. Since the zero-pressure Newtonian perturbation equations have only quadratic order nonlinearity, any non-vanishing third-order terms in relativistic analysis can be regarded as a pure general relativistic correction terms. In [4] we presented such third-order correction terms in a single component case. In the next section we will derive the third-order correction terms appearing in the multi-component situation. As the third-order analysis closely follows the case of second-order, in the following we will derive the second-order equations directly from the ADM equations presented in Sec. II A.

We consider zero-pressure multi-fluids, thus

$$p_{(i)} \equiv 0, \quad \delta p_{(i)} \equiv 0 \equiv \Pi_{(i)\alpha\beta}. \tag{47}$$

The irrotational condition and the temporal comoving gauge condition lead to $v_\alpha = 0$, thus we have $J_\alpha = 0$; as the spatial gauge condition we take $\gamma \equiv 0$, thus $\beta = \chi/a$. To the second-order perturbations, Eq. (12) gives

$$\alpha = -\frac{1}{2a^2}\chi^{,\alpha}\chi_{,\alpha} - \sum_j \frac{\mu_{(j)}}{\mu}\left[\frac{1}{2}v^\alpha_{(j)}v_{(j)\alpha} + \Delta^{-1}\nabla_\alpha\left(v^\alpha_{(j)}v^\beta_{(j)|\beta}\right)\right], \tag{48}$$

thus

$$N = a - a\sum_j \frac{\mu_{(j)}}{\mu}\left[\frac{1}{2}v^\alpha_{(j)}v_{(j)\alpha} + \Delta^{-1}\nabla_\alpha\left(v^\alpha_{(j)}v^\beta_{(j)|\beta}\right)\right]. \tag{49}$$

Using Eqs. (45),(46), Eqs. (11),(13),(14),(15) give

$$\dot{\delta} - \kappa = -\frac{c}{a^2}\delta_{,\alpha}\chi^{,\alpha} + \delta\kappa + \frac{1}{2}H\sum_j \mu_{(j)}v^\alpha_{(j)}v_{(j)\alpha} + 3H\sum_j \frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_\alpha\left(v^\alpha_{(j)}v^\beta_{(j)|\beta}\right), \tag{50}$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\varrho\delta = -\frac{c}{a^2}\kappa_{,\alpha}\chi^{,\alpha} + \frac{1}{3}\kappa^2 + \left(\dot{C}^{(t)\alpha\beta} + \frac{c}{a^2}\chi^{,\alpha|\beta}\right)\left(\dot{C}^{(t)}_{\alpha\beta} + \frac{c}{a^2}\chi_{,\alpha|\beta}\right) - \frac{1}{3}\left(c\frac{\Delta}{a^2}\chi\right)^2$$

$$+\frac{1}{2}\left(3\dot{H} + 8\pi G\varrho + c^2\frac{\Delta}{a^2}\right)\sum_j \mu_{(j)}v^\alpha_{(j)}v_{(j)\alpha} + \left(3\dot{H} + c^2\frac{\Delta}{a^2}\right)\sum_j \frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_\alpha\left(v^\alpha_{(j)}v^\beta_{(j)|\beta}\right), \tag{51}$$

$$\dot{\delta}_{(i)} - \kappa + c\frac{1}{a}\left[\left(1 + \delta_{(i)}\right)v^\alpha_{(i)}\right]_{|\alpha} = -\frac{c}{a^2}\delta_{(i),\alpha}\chi^{,\alpha} + \delta_{(i)}\kappa + Hv_{(i)\alpha}v^\alpha_{(i)}$$

$$-\frac{c}{a}\left(\varphi^{,\alpha}v_{(i)\alpha} - 2\varphi v^\alpha_{(i)|\alpha} - 2v^{\alpha|\beta}_{(i)}C^{(t)}_{\alpha\beta}\right) + \frac{3}{2}H\sum_j \mu_{(j)}v^\alpha_{(j)}v_{(j)\alpha} + 3H\sum_j \frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_\alpha\left(v^\alpha_{(j)}v^\beta_{(j)|\beta}\right), \tag{52}$$

$$\frac{1}{a}\left[a\left(1 + \delta_{(i)}\right)v_{(i)\alpha}\right]^{\cdot} = -\frac{c}{a^2}\left(v_{(i)\beta}\chi^{,\beta}\right)_{,\alpha} + \kappa v_{(i)\alpha} - \frac{c}{a}\left(v_{(i)\alpha}v^\beta_{(i)}\right)_{|\beta}$$

$$+\frac{c}{2a}\sum_j \mu_{(j)}\nabla_\alpha\left(v^\beta_{(j)}v_{(j)\beta}\right) + \frac{c}{a}\sum_j \frac{\mu_{(j)}}{\mu}\nabla_\alpha\Delta^{-1}\nabla_\gamma\left(v^\gamma_{(j)}v^\beta_{(j)|\beta}\right). \tag{53}$$

From Eqs. (50),(51), and Eqs. (51)-(53) we can derive

$$\frac{1}{a^2}\left[a^2\left(\dot{\delta}+\frac{c}{a^2}\delta_{,\alpha}\chi^{,\alpha}\right)\right]^{\cdot}-4\pi G\varrho\delta\left(1+\delta\right)=-\frac{c}{a^2}\kappa_{,\alpha}\chi^{,\alpha}+\frac{4}{3}\kappa^2$$

$$+\left(\dot{C}^{(t)\alpha\beta}+\frac{c}{a^2}\chi^{,\alpha|\beta}\right)\left(\dot{C}^{(t)}_{\alpha\beta}+\frac{c}{a^2}\chi_{,\alpha|\beta}\right)-\frac{1}{3}\left(c\frac{\Delta}{a^2}\chi\right)^2$$

$$+\left(2\dot{H}+4\pi G\varrho+\frac{1}{2}c^2\frac{\Delta}{a^2}\right)\sum_j\mu_{(j)}v^{\alpha}_{(j)}v_{(j)\alpha}+\left(6\dot{H}+c^2\frac{\Delta}{a^2}\right)\sum_j\frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_{\alpha}\left(v^{\alpha}_{(j)}v^{\beta}_{(j)|\beta}\right),\qquad(54)$$

$$\frac{1}{a^2}\left[a^2\left(\dot{\delta}_{(i)}+\frac{c}{a^2}\delta_{(i),\alpha}\chi^{,\alpha}\right)\right]^{\cdot}-4\pi G\varrho\delta\left(1+\delta_{(i)}\right)=-\frac{c}{a^2}\kappa_{,\alpha}\chi^{,\alpha}+\frac{4}{3}\kappa^2$$

$$+\left(\dot{C}^{(t)\alpha\beta}+\frac{c}{a^2}\chi^{,\alpha|\beta}\right)\left(\dot{C}^{(t)}_{\alpha\beta}+\frac{c}{a^2}\chi_{,\alpha|\beta}\right)-\frac{1}{3}\left(c\frac{\Delta}{a^2}\chi\right)^2-\frac{c}{a}\left[\left(\kappa+\dot{\varphi}\right)^{,\alpha}v_{(i)\alpha}+2\left(\kappa-\dot{\varphi}\right)v^{\alpha}_{(i)|\alpha}\right]$$

$$+c^2\frac{\Delta}{a^3}\left(v_{(i)\alpha}\chi^{,\alpha}\right)+\frac{c^2}{a^2}\left(v^{\alpha}_{(i)}v^{\beta}_{(i)}\right)_{|\alpha\beta}+\dot{H}v_{(i)\alpha}v^{\alpha}_{(i)}+\frac{2c}{a}v^{\alpha|\beta}_{(i)}\dot{C}^{(t)}_{\alpha\beta}$$

$$+\left(3\dot{H}+4\pi G\varrho\right)\sum_j\mu_{(j)}v^{\alpha}_{(j)}v_{(j)\alpha}+6\dot{H}\sum_j\frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_{\alpha}\left(v^{\alpha}_{(j)}v^{\beta}_{(j)|\beta}\right).\qquad(55)$$

We have recovered c using

$$[G\varrho]=T^{-2},\quad[p]=[\mu]=[\varrho c^2],\quad[a]=L,\quad[R^{(3)}]=1,\quad[\nabla]=1,$$

$$[\varphi]=[\beta]=[\gamma]=[C^{(t)}_{\alpha\beta}]=[v_{\alpha}]=[v_{(i)\alpha}]=[\delta]=[\delta_{(i)}]=1,\quad[\chi]=L,\quad[\kappa]=T^{-1},\qquad(56)$$

where $R^{(3)}=6\bar{K}$ with normalized $\bar{K}=0$ or ±1.

Without the rotational mode, we introduce

$$v_{(i)\alpha}\equiv-v_{(i),\alpha}.\qquad(57)$$

Since we used the comoving gauge condition, to the linear order, we have

$$v_{(i)}=v_{(i)v}\equiv v_{(i)}-v=v_{(i)\chi}-v_{\chi},\qquad(58)$$

where $v_{\chi}\equiv v-\chi/a$ and $v_{(i)\chi}\equiv v_{(i)}-\chi/a$ to the linear order. To the linear order, Newtonian velocity perturbation variables are introduced as [17]

$$\mathbf{u}\equiv\nabla u\equiv-c\nabla v_{\chi},\quad\mathbf{u}_i\equiv\nabla u_i\equiv-c\nabla v_{(i)\chi},\qquad(59)$$

thus

$$\mathbf{v}_{(i)}\equiv-\nabla v_{(i)}=\frac{1}{c}\left(\mathbf{u}_i-\mathbf{u}\right).\qquad(60)$$

Dimensions are

$$[v]=[v_{(i)}]=1,\quad[\mathbf{u}]=[\mathbf{u}_i]=[u]=[u_i]=[c]=L/T.\qquad(61)$$

Thus, we have

$$\frac{\delta p}{\mu}=\frac{1}{3}\sum_j\frac{\varrho_j}{\varrho}\frac{1}{c^2}|\mathbf{u}_j-\mathbf{u}|^2,$$

$$\sum_j\frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_{\alpha}\left(v^{\alpha}_{(j)}v^{\beta}_{(j)|\beta}\right)=\sum_j\frac{\varrho_j}{\varrho}\frac{1}{c^2}\Delta^{-1}\nabla\cdot\left[\left(\mathbf{u}_j-\mathbf{u}\right)\nabla\cdot\left(\mathbf{u}_j-\mathbf{u}\right)\right].\qquad(62)$$

A. Newtonian correspondence

We *assume* a flat background, thus $R^{(3)}=0$. Thus, we have $\dot{\varphi}_v=0$ to the linear order even in the presence of the cosmological constant and multiple components [17, 18], see Eq. (99). To the second order, we *identify* the Newtonian perturbation variables δ, δ_i, $\mathbf{u}$, and $\mathbf{u}_i$ as

$$\kappa_v\equiv-\frac{1}{a}\nabla\cdot\mathbf{u},\quad\mathbf{u}\equiv\nabla u,\quad\mathbf{v}_{(i)v}\equiv\frac{1}{c}\left(\mathbf{u}_i-\mathbf{u}\right),\quad\delta\equiv\delta_v,\quad\delta_i\equiv\delta_{(i)v}.\qquad(63)$$

Comparison of the consequent relativistic equations with the Newtonian equations will show apparently why these identifications are the proper ones. Examination of Eqs. (50)-(55) shows that we need $\chi(= \chi_v)$ only to the linear order. From the momentum constraint equation in Eq. (16), see Eq. (197) of [1], we have

$$\kappa_v = -c\frac{\Delta}{a^2}\chi_v, \tag{64}$$

thus

$$\chi_v \equiv \frac{a}{c}u, \tag{65}$$

to the linear order.

Equations (50)-(55) become

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta\mathbf{u}) + H\sum_j \frac{\varrho_j}{\varrho}\frac{1}{c^2}\left\{\frac{1}{2}|\mathbf{u}_j - \mathbf{u}|^2 + 3\Delta^{-1}\nabla \cdot [(\mathbf{u}_j - \mathbf{u})\nabla \cdot (\mathbf{u}_j - \mathbf{u})]\right\}, \tag{66}$$

$$\frac{1}{a}\nabla \cdot (\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right)$$

$$+ \sum_j \frac{\varrho_j}{\varrho}\frac{1}{c^2}\left\{\frac{1}{2}\left(4\pi G\varrho - c^2\frac{\Delta}{a^2}\right)|\mathbf{u}_j - \mathbf{u}|^2 + \left(12\pi G\varrho - c^2\frac{\Delta}{a^2}\right)\Delta^{-1}\nabla \cdot [(\mathbf{u}_j - \mathbf{u})\nabla \cdot (\mathbf{u}_j - \mathbf{u})]\right\}, \tag{67}$$

$$\dot{\delta}_i + \frac{1}{a}\nabla \cdot \mathbf{u}_i = -\frac{1}{a}\nabla \cdot (\delta_i\mathbf{u}_i) + \frac{1}{a}\left[2\varphi\nabla \cdot (\mathbf{u}_i - \mathbf{u}) - (\mathbf{u}_i - \mathbf{u}) \cdot \nabla\varphi + 2(u_i^\alpha - u^\alpha)^{|\beta}C^{(t)}_{\alpha\beta}\right]$$

$$+ H\frac{1}{c^2}|\mathbf{u}_i - \mathbf{u}|^2 + 3H\sum_j \frac{\varrho_j}{\varrho}\frac{1}{c^2}\left\{\frac{1}{2}|\mathbf{u}_j - \mathbf{u}|^2 + \Delta^{-1}\nabla \cdot [(\mathbf{u}_j - \mathbf{u})\nabla \cdot (\mathbf{u}_j - \mathbf{u})]\right\}, \tag{68}$$

$$\frac{1}{a}\nabla \cdot (\dot{\mathbf{u}}_i + H\mathbf{u}_i) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u}_i \cdot \nabla\mathbf{u}_i) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right)$$

$$+ 4\pi G\sum_j \varrho_j\frac{1}{c^2}\left\{\frac{1}{2}|\mathbf{u}_j - \mathbf{u}|^2 + 3\Delta^{-1}\nabla \cdot [(\mathbf{u}_j - \mathbf{u})\nabla \cdot (\mathbf{u}_j - \mathbf{u})]\right\}, \tag{69}$$

$$\frac{1}{a^2}\left(a^2\dot{\delta}\right)^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}[a\nabla \cdot (\delta\mathbf{u})]^{\cdot} + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right)$$

$$- \sum_j \frac{\varrho_j}{\varrho}\frac{1}{c^2}\left\{\left(4\pi G\varrho - \frac{c^2}{2}\frac{\Delta}{a^2}\right)|\mathbf{u}_j - \mathbf{u}|^2 + \left(24\pi G\varrho - c^2\frac{\Delta}{a^2}\right)\Delta^{-1}\nabla \cdot [(\mathbf{u}_j - \mathbf{u})\nabla \cdot (\mathbf{u}_j - \mathbf{u})]\right\}, \tag{70}$$

$$\frac{1}{a^2}\left(a^2\dot{\delta}_i\right)^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}[a\nabla \cdot (\delta_i\mathbf{u}_i)]^{\cdot} + \frac{1}{a^2}\nabla \cdot (\mathbf{u}_i \cdot \nabla\mathbf{u}_i) + \dot{C}^{(t)}_{\alpha\beta}\left(\frac{2}{a}u_i^{\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right)$$

$$+ \frac{1}{a^2}\{\Delta[\mathbf{u} \cdot (\mathbf{u}_i - \mathbf{u})] - \nabla \cdot [(\mathbf{u}_i - \mathbf{u}) \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla(\mathbf{u}_i - \mathbf{u})]\}$$

$$- \frac{4\pi G\varrho}{c^2}|\mathbf{u}_i - \mathbf{u}|^2 - 8\pi G\sum_j \varrho_j\frac{1}{c^2}\{|\mathbf{u}_j - \mathbf{u}|^2 + 3\Delta^{-1}\nabla \cdot [(\mathbf{u}_j - \mathbf{u})\nabla \cdot (\mathbf{u}_j - \mathbf{u})]\}. \tag{71}$$

In [5] we have shown that, to the linear order, $(\mathbf{u}_i - \mathbf{u})$ simply decays

$$\mathbf{u}_i - \mathbf{u} \propto \frac{1}{a}, \tag{72}$$

in an expanding phase. Thus, ignoring quadratic combination of $(\mathbf{u}_i - \mathbf{u})$ terms, we have

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta\mathbf{u}), \tag{73}$$

$$\frac{1}{a}\nabla \cdot (\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{74}$$

$$\frac{1}{a^2}\left(a^2\dot{\delta}\right)^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}[a\nabla \cdot (\delta\mathbf{u})]^{\cdot} + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{75}$$

and

$$\dot{\delta}_i + \frac{1}{a}\nabla \cdot \mathbf{u}_i = -\frac{1}{a}\nabla \cdot (\delta_i \mathbf{u}_i) + \frac{1}{a}\left[2\varphi\nabla \cdot (\mathbf{u}_i - \mathbf{u}) - (\mathbf{u}_i - \mathbf{u}) \cdot \nabla\varphi + 2\left(u_i^\alpha - u^\alpha\right)^{|\beta} C_{\alpha\beta}^{(t)}\right], \tag{76}$$

$$\frac{1}{a}\nabla \cdot (\dot{\mathbf{u}}_i + H\mathbf{u}_i) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u}_i \cdot \nabla\mathbf{u}_i) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}_{\alpha\beta}^{(t)}\right), \tag{77}$$

$$\frac{1}{a^2}\left(a^2\dot{\delta}_i\right)^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}\left[a\nabla \cdot (\delta_i\mathbf{u}_i)\right]^{\cdot} + \frac{1}{a^2}\nabla \cdot (\mathbf{u}_i \cdot \nabla\mathbf{u}_i) + \dot{C}_{\alpha\beta}^{(t)}\left(\frac{2}{a}u_i^{\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right)$$

$$+\frac{1}{a^2}\left\{\Delta\left[\mathbf{u} \cdot (\mathbf{u}_i - \mathbf{u})\right] - \nabla \cdot \left[(\mathbf{u}_i - \mathbf{u}) \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla(\mathbf{u}_i - \mathbf{u})\right]\right\}. \tag{78}$$

Equations (73)-(75) coincide with the density and velocity perturbation equations of a single component medium [2]; thus, except for the contribution from gravitational waves, these equations coincide with ones in the Newtonian context. In the Newtonian context, Eqs. (73)-(75) without the gravitational waves were presented in [11] which are valid to fully nonlinear order. To the linear order, Eq. (75) was derived by Lifshitz [9] in the synchronous gauge, and by Nariai [19] in the comoving gauge. In the zero-pressure medium the free-falling object is also comoving, thus we can impose both the synchronous gauge and the comoving gauge simultaneously. In [3] we compared subtle differences of the second-order perturbation equations in the synchronous gauge with the ones in the comoving gauge.

If we further ignore $(\mathbf{u}_i - \mathbf{u})$ terms appearing in the pure second-order combinations, Eqs. (76)-(78) become

$$\dot{\delta}_i + \frac{1}{a}\nabla \cdot \mathbf{u}_i = -\frac{1}{a}\nabla \cdot (\delta_i \mathbf{u}_i), \tag{79}$$

$$\frac{1}{a}\nabla \cdot (\dot{\mathbf{u}}_i + H\mathbf{u}_i) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u}_i \cdot \nabla\mathbf{u}_i) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}_{\alpha\beta}^{(t)}\right), \tag{80}$$

$$\frac{1}{a^2}\left(a^2\dot{\delta}_i\right)^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}\left[a\nabla \cdot (\delta_i\mathbf{u}_i)\right]^{\cdot} + \frac{1}{a^2}\nabla \cdot (\mathbf{u}_i \cdot \nabla\mathbf{u}_i) + \dot{C}_{\alpha\beta}^{(t)}\left(\frac{2}{a}u_i^{\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right), \tag{81}$$

which coincide with the Newtonian equations except for the contributions from the gravitational waves. In this context, except for the contribution from gravitational waves, the above equations coincide with ones in the Newtonian context even in the multi-component case. Therefore, we have shown the relativistic/Newtonian correspondence, except for the contributions from the gravitational waves, to the second-order perturbations in the case of multi-component, zero-pressure, irrotational fluids assuming a flat background.

IV. THIRD-ORDER EQUATIONS

We *consider* irrotational fluids, thus ignore all vector-type perturbations. As all three-types of perturbations are generally coupled in nonlinear perturbations, apparently this is an important assumption we make in this section. In an expanding phase, however, the linear order rotational perturbations have only decaying mode due to the angular-momentum conservation. Effects of rotational perturbations to the second-order perturbations are considered in our accompanying work in [5].

We *take* the temporal comoving gauge

$$v \equiv 0. \tag{82}$$

Together with the irrotational condition $v_\alpha^{(v)} \equiv 0$ we have $v_\alpha = 0$. Equation (34) shows that this leads to $J_\alpha = 0$ for general fluids. As the spatial gauge condition we *take*

$$\gamma \equiv 0. \tag{83}$$

In [1] we have shown that these gauge conditions fix the space-time gauge transformation properties completely to all orders in perturbations. Thus, each perturbation variable under these gauge conditions has a corresponding unique gauge-invariant combination, and can be equivalently regarded as *gauge-invariant* one to all orders in perturbations.

We consider zero-pressure fluids, thus

$$p_{(i)} = 0, \quad \delta p_{(i)} \equiv 0 \equiv \Pi_{(i)\alpha\beta}. \tag{84}$$

The collective fluid quantities are nontrivial and are presented in Eq. (44).

In our previous study of second-order perturbations, summarized in a previous section, we showed that $\mathcal{O}(|v_{(i)\alpha}|^2)$ terms simply correspond to pure decaying mode in expanding phase. We showed that by ignoring these terms we recover complete relativistic/Newtonian correspondence even in the multi-component case. Based on this observation, in our third order calculation in the following we will *ignore* $\mathcal{O}(|v_{(i)\alpha}|^2)$ terms. If we *ignore* $\mathcal{O}(|v_{(i)\alpha}|^2)$ terms, the collective fluid quantities in Eq. (44) give

$$\delta\mu = \sum_j \delta\mu_{(j)}, \quad \delta p = 0, \quad 0 = \sum_j \left(\mu_{(j)} v_{(j)\alpha} + \delta\mu_{(j)} v_{(j)\alpha} \right), \quad \Pi_\beta^\alpha = 0, \tag{85}$$

and the ADM fluid quantities in Eqs. (45),(46) become

$$E = \mu + \delta\mu, \quad J_\alpha = 0, \quad S_{\alpha\beta} = 0, \quad S = 0, \quad \bar{S}_{\alpha\beta} = 0,$$
$$E_{(i)} = \mu_{(i)} + \delta\mu_{(i)}, \quad J_{(i)\alpha} = a\left(\mu_{(i)} v_{(i)\alpha} + \delta\mu_{(i)} v_{(i)\alpha} \right), \quad S_{(i)\alpha\beta} = 0, \quad S_{(i)} = 0, \quad \bar{S}_{(i)\alpha\beta} = 0. \tag{86}$$

Equation (12) gives

$$N = a(t), \tag{87}$$

thus,

$$\alpha = -\frac{1}{2a^2}\chi^{,\alpha}\chi_{,\alpha}\left(1 - 2\varphi\right) + \frac{1}{a^2}\chi^{,\alpha}\chi^{,\beta}C_{\alpha\beta}^{(t)}. \tag{88}$$

Using Eq. (86), Eqs. (11),(13),(14),(15) give

$$\left(\frac{\dot{\mu}}{\mu} + 3H \right)(1 + \delta) + \dot{\delta} - \kappa = -\frac{c}{a^2}\delta_{,\alpha}\chi^{,\alpha}\left(1 - 2\varphi\right) + \kappa\delta + \frac{2c}{a^2}\delta^{,\alpha}\chi^{,\beta}C_{\alpha\beta}^{(t)}, \tag{89}$$

$$-\left[3\dot{H} + 3H^2 + 4\pi G\varrho - \Lambda c^2 \right] + \dot{\kappa} + 2H\kappa - 4\pi G\varrho\delta = -\frac{c}{a^2}\kappa_{,\alpha}\chi^{,\alpha}\left(1 - 2\varphi\right) + \frac{1}{3}\kappa^2 + \frac{2c}{a^2}\kappa^{,\alpha}\chi^{,\beta}C_{\alpha\beta}^{(t)}$$

$$+ \left(\frac{c}{a^2}\chi^{,\alpha|\beta} - \frac{c}{3}g^{(3)\alpha\beta}\frac{\Delta}{a^2}\chi + \dot{C}^{(t)\alpha\beta} \right)\left[\left(\frac{c}{a^2}\chi_{,\alpha|\beta} + \dot{C}_{\alpha\beta}^{(t)} \right)(1 - 4\varphi) \right.$$

$$\left. - \frac{4c}{a^2}\chi_{,\alpha}\varphi_{,\beta} - \frac{2c}{a^2}\chi^{,\gamma}\left(2C_{\gamma\alpha|\beta}^{(t)} - C_{\alpha\beta|\gamma}^{(t)} \right) - 4C_{\beta\gamma}^{(t)}\left(\frac{c}{a^2}\chi^{,\gamma}_{\ |\alpha} + \dot{C}^{(t)\gamma}_{\ \alpha} \right) - 4\dot{\varphi}C_{\alpha\beta}^{(t)} \right], \tag{90}$$

$$\left(\frac{\dot{\mu}_{(i)}}{\mu_{(i)}} + 3H \right)\left(1 + \delta_{(i)}\right) + \dot{\delta}_{(i)} - \kappa + \frac{c}{a}\left[\left(1 + \delta_{(i)}\right)v_{(i)}^\alpha \right]_{|\alpha}\left(1 - 2\varphi + 4\varphi^2 \right)$$

$$= -\frac{c}{a^2}\delta_{(i),\alpha}\chi^{,\alpha}\left(1 - 2\varphi\right) + \kappa\delta_{(i)} - \frac{c}{a}\varphi_{,\alpha}v_{(i)}^\alpha\left(1 - 4\varphi\right) + \frac{2c}{a}C_{\alpha\beta}^{(t)}v_{(i)}^{\alpha|\beta}\left(1 - 4\varphi\right) + \frac{2c}{a^2}\delta_{(i)}^{,\alpha}\chi^{,\beta}C_{\alpha\beta}^{(t)}$$

$$+ \frac{c}{a}\left[-\varphi^{,\alpha}\left(\delta_{(i)}v_{(i)\alpha} + 2C_{\alpha\beta}^{(t)}v_{(i)}^\beta \right) + 2C^{(t)\beta\gamma}C_{\beta\gamma|\alpha}^{(t)}v_{(i)}^\alpha + 2\left(\delta_{(i)}C^{(t)\alpha\beta}v_{(i)\beta} - 2C^{(t)\alpha\beta}C_{\beta\gamma}^{(t)}v_{(i)}^\gamma \right)_{|\alpha} \right], \tag{91}$$

$$\frac{1}{a}\left[a\left(1 + \delta_{(i)}\right)v_{(i)\alpha} \right]^{\cdot} = \left(1 + \delta_{(i)}\right)\kappa v_{(i)\alpha}$$

$$- \frac{c}{a^2}\left\{ \left(v_{(i)\beta}\chi^{,\beta}\right)_{,\alpha}\left(1 + \delta_{(i)}\right) + \delta_{(i),\beta}\chi^{,\beta}v_{(i)\alpha} - 2\left[v_{(i)}^\beta\left(\chi_{,\beta}\varphi + \chi^{,\gamma}C_{\beta\gamma}^{(t)} \right) \right]_{,\alpha} \right\}, \tag{92}$$

where we recovered c. Equation (92) can be written as

$$\frac{1}{a}\left(av_{(i)\alpha} \right)^{\cdot} = -\frac{c}{a^2}\left[v_{(i)\beta}\chi^{,\beta}\left(1 - 2\varphi\right) - 2v_{(i)}^\beta\chi^{,\gamma}C_{\beta\gamma}^{(t)} \right]_{,\alpha}. \tag{93}$$

Equations (89) and (90) are the same as Eqs. (20) and (21) in [4] which were derived in the single component situation.

Combining Eqs. (89),(90) and Eqs. (90)-(92), respectively, we can derive

$$\frac{1}{a^2}\left[a^2\dot\delta + c\delta_{,\alpha}\chi^{,\alpha}(1-2\varphi) - 2c\delta^{,\alpha}\chi^{,\beta}C_{\alpha\beta}^{(t)}\right]^{\cdot} - 4\pi G\varrho\delta(1+\delta)$$

$$= \frac{4}{3}\kappa^2(1+\delta) - \frac{c}{a^2}\left[\kappa_{,\alpha}\chi^{,\alpha}(1-2\varphi) + (\delta\kappa)_{,\alpha}\chi^{,\alpha} - 2\kappa^{,\alpha}\chi^{,\beta}C_{\alpha\beta}^{(t)}\right]$$

$$+ \left(\frac{c}{a^2}\chi^{,\alpha|\beta} - \frac{c}{3}g^{(3)\alpha\beta}\frac{\Delta}{a^2}\chi + \dot{C}^{(t)\alpha\beta}\right)\left[\left(\frac{c}{a^2}\chi_{,\alpha|\beta} + \dot{C}_{\alpha\beta}^{(t)}\right)(1-4\varphi+\delta)\right.$$

$$\left. - \frac{4c}{a^2}\chi_{,\alpha}\varphi_{,\beta} - \frac{2c}{a^2}\chi^{,\gamma}\left(2C_{\gamma\alpha|\beta}^{(t)} - C_{\alpha\beta|\gamma}^{(t)}\right) - 4C_{\beta\gamma}^{(t)}\left(\frac{c}{a^2}\chi^{,\gamma}{}_{|\alpha} + \dot{C}^{(t)\gamma}{}_{\alpha}\right) - 4\dot\varphi C_{\alpha\beta}^{(t)}\right], \tag{94}$$

$$\frac{1}{a^2}\left[a^2\dot\delta_{(i)} + c\delta_{(i),\alpha}\chi^{,\alpha}(1-2\varphi) - 2c\delta_{(i)}^{,\alpha}\chi^{,\beta}C_{\alpha\beta}^{(t)}\right]^{\cdot} - 4\pi G\varrho\delta\left(1+\delta_{(i)}\right)$$

$$- \frac{4}{3}\kappa^2\left(1+\delta_{(i)}\right) + \frac{c}{a^2}\left[\kappa_{,\alpha}\chi^{,\alpha}(1-2\varphi) + \left(\delta_{(i)}\kappa\right)_{,\alpha}\chi^{,\alpha} - 2\kappa^{,\alpha}\chi^{,\beta}C_{\alpha\beta}^{(t)}\right]$$

$$- \left(\frac{c}{a^2}\chi^{,\alpha|\beta} - \frac{c}{3}g^{(3)\alpha\beta}\frac{\Delta}{a^2}\chi + \dot{C}^{(t)\alpha\beta}\right)\left[\left(\frac{c}{a^2}\chi_{,\alpha|\beta} + \dot{C}_{\alpha\beta}^{(t)}\right)\left(1-4\varphi+\delta_{(i)}\right)\right.$$

$$\left. - \frac{4c}{a^2}\chi_{,\alpha}\varphi_{,\beta} - \frac{2c}{a^2}\chi^{,\gamma}\left(2C_{\gamma\alpha|\beta}^{(t)} - C_{\alpha\beta|\gamma}^{(t)}\right) - 4C_{\beta\gamma}^{(t)}\left(\frac{c}{a^2}\chi^{,\gamma}{}_{|\alpha} + \dot{C}^{(t)\gamma}{}_{\alpha}\right) - 4\dot\varphi C_{\alpha\beta}^{(t)}\right]$$

$$= \frac{c}{a}\left\{-2(\kappa-\dot\varphi)v_{(i)|\alpha}^{\alpha} - (\kappa+\dot\varphi)_{,\alpha}v_{(i)}^{\alpha} + c\frac{\Delta}{a^2}\left(v_{(i)\alpha}\chi^{,\alpha}\right) + 2v_{(i)}^{\alpha|\beta}\dot{C}_{\alpha\beta}^{(t)}\right.$$

$$+ v_{(i)}^{\alpha}\left[(2\varphi-\delta_{(i)})\kappa_{,\alpha} - (\varphi+2\delta_{(i)})_{,\alpha}\kappa + 2\dot\varphi\delta_{(i),\alpha}\right.$$

$$\left. + \left(4\varphi\varphi_{,\alpha} - \varphi_{,\alpha}\delta_{(i)} - 2\varphi^{,\beta}C_{\alpha\beta}^{(t)} + 2C^{(t)\beta\gamma}C_{\beta\gamma|\alpha}^{(t)} + 2\delta_{(i)}^{;\beta}C_{\alpha\beta}^{(t)} - 4C^{(t)\beta\gamma}C_{\gamma\alpha|\beta}^{(t)}\right)^{\cdot}\right]$$

$$\left. + 2v_{(i)}^{\alpha|\beta}\left[\kappa C_{\alpha\beta}^{(t)} + \left(-4\varphi C_{\alpha\beta}^{(t)} + \delta_{(i)}C_{\alpha\beta}^{(t)} - 2C^{(t)\gamma}{}_{\beta}C_{\gamma\alpha}^{(t)}\right)^{\cdot}\right] + 2v_{(i)|\alpha}^{\alpha}\left[\kappa\left(2\varphi-\delta_{(i)}\right) + \dot\varphi\left(\delta_{(i)} - 4\varphi\right)\right]\right\}$$

$$+ \frac{c^2}{a^3}\left\{\left(\delta_{(i)} - 2\varphi\right)\Delta\left(v_{(i)\alpha}\chi^{,\alpha}\right) + \left(v_{(i)\beta}\chi^{,\beta}\right)_{,\alpha}\delta_{(i)}^{;\alpha} + \left(\delta_{(i),\beta}\chi^{,\beta}v_{(i)\alpha}\right)^{|\alpha}\right.$$

$$\left. - 2\Delta\left[v_{(i)}^{\alpha}\left(\chi_{,\alpha}\varphi + \chi^{,\beta}C_{\alpha\beta}^{(t)}\right)\right] + \left(v_{(i)\beta}\chi^{,\beta}\right)^{,\alpha}\varphi_{,\alpha} - 2\left(v_{(i)\gamma}\chi^{,\gamma}\right)^{,\alpha|\beta}C_{\alpha\beta}^{(t)}\right\}. \tag{95}$$

All terms in the RHS of Eq. (95) contain $v_{(i)}^{\alpha}$ which decays to the linear order as

$$\mathbf{v}_{(i)} = \frac{1}{c}\left(\mathbf{u}_i - \mathbf{u}\right) \propto \frac{1}{a}, \tag{96}$$

see Eqs. (60),(72) and [5]. By setting $\delta_{(i)} = \delta$ the LHS of Eq. (95) is the same as Eq. (94). By setting $\delta_{(i)} = \delta$ and $v_{(i)}^{\alpha} = 0$, Eq. (95) becomes Eq. (94).

In order to complete Eq. (95) we need equations for $\dot\varphi$, $v_{(i)}^{\alpha}$, and $C_{\alpha\beta}^{(t)}$ terms to the second-order perturbations. Equation for $v_{(i)}^{\alpha}$ is in Eq. (93) which gives

$$\frac{1}{a}\left(av_{(i)\alpha}\right)^{\cdot} = -\frac{c}{a^2}\left(v_{(i)\beta}\chi^{,\beta}\right)_{,\alpha}, \tag{97}$$

to the second order. The relation between κ and χ can be found in Eq. (23) of [4]. Recovering the background

curvature, from the momentum constraint equation in Eq. (16), we can derive

$$\kappa + \frac{c}{a^2}\left(\Delta + \frac{R^{(3)}}{2}\right)\chi = \frac{c}{a^2}\left(2\varphi\Delta\chi - \chi^{,\alpha}\varphi_{,\alpha}\right) + C^{(t)\alpha\beta}\left(\frac{2c}{a^2}\chi_{,\alpha|\beta} - \dot{C}^{(t)}_{\alpha\beta}\right)$$

$$+ \frac{3}{2}\Delta^{-1}\nabla^\alpha\left\{\frac{c}{a^2}\left[\chi^{,\beta}\varphi_{,\alpha|\beta} + \chi_{,\alpha}\left(\Delta + \frac{2}{3}R^{(3)}\right)\varphi\right]\right.$$

$$\left. + \chi^{,\beta}\frac{c}{a^2}\left(\Delta - \frac{R^{(3)}}{3}\right)C^{(t)}_{\alpha\beta} - \varphi^{,\beta}\dot{C}^{(t)}_{\alpha\beta} + 2C^{(t)\beta\gamma}\dot{C}^{(t)}_{\alpha\beta|\gamma} + C^{(t)}_{\beta\gamma|\alpha}\dot{C}^{(t)\beta\gamma}\right\}, \tag{98}$$

to the second order. Equation for φ to the second order follows from Eq. (99) in [1]. Ignoring $\mathcal{O}(|v_{(i)\alpha}|^2)$, thus using Eq. (88) for α we can derive

$$3\dot{\varphi} = -\left(\kappa + c\frac{\Delta}{a^2}\chi\right) + \frac{c}{a^2}\left(2\varphi\Delta\chi - \chi^{,\alpha}\varphi_{,\alpha}\right) + 2C^{(t)\alpha\beta}\left(\frac{c}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{99}$$

In a flat background, the RHS is the second order; thus, to the linear order we have $\dot{\varphi} = 0$. Equation for $C^{(t)}_{\alpha\beta}$ to the second order is presented in Eq. (43) of [4].

A. Pure general relativistic corrections

Now, we *assume*

$$R^{(3)} = 0, \tag{100}$$

thus $\dot{\varphi} = 0$ to the linear order. Based on the apparent success in second-order perturbations, we continue to use the *identifications* made in Eq. (63) valid to the third order, thus

$$\kappa_v \equiv -\frac{1}{a}\nabla \cdot \mathbf{u}, \quad \mathbf{u} \equiv \nabla u, \quad \mathbf{v}_{(i)v} \equiv \frac{1}{c}\left(\mathbf{u}_i - \mathbf{u}\right), \quad \delta \equiv \delta_v, \quad \delta_i \equiv \delta_{(i)v}. \tag{101}$$

In the following we consider pure scalar-type perturbations, thus set $C^{(t)}_{\alpha\beta} \equiv 0$. Contributions from the gravitational waves will be considered in the next section.

We need χ_v only to the second order. Equation (98) gives

$$\chi_v \equiv \frac{a}{c}\left(u + \Delta^{-1}X\right), \tag{102}$$

where

$$X \equiv 2\varphi\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla\left[\mathbf{u} \cdot \nabla\left(\nabla\varphi\right) + \mathbf{u}\Delta\varphi\right]. \tag{103}$$

Equations (89)-(92) become

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta\mathbf{u}) + \frac{1}{a}\left(2\varphi\mathbf{u} - \nabla\Delta^{-1}X\right) \cdot \nabla\delta, \tag{104}$$

$$\frac{1}{a}\nabla \cdot (\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) - \frac{\Delta}{a^2}\left[\mathbf{u} \cdot \nabla\left(\Delta^{-1}X\right)\right] + \frac{1}{a^2}\left(\mathbf{u} \cdot \nabla X + \frac{2}{3}X\nabla \cdot \mathbf{u}\right)$$

$$-\frac{2}{3a^2}\varphi\mathbf{u} \cdot \nabla\left(\nabla \cdot \mathbf{u}\right) + \frac{4}{a^2}\nabla \cdot \left[\varphi\left(\mathbf{u} \cdot \nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla \cdot \mathbf{u}\right)\right], \tag{105}$$

$$\dot{\delta}_i + \frac{1}{a}\nabla \cdot \mathbf{u}_i = -\frac{1}{a}\nabla \cdot (\delta_i\mathbf{u}_i) + \frac{1}{a}\left[2\varphi\mathbf{u}_i - \nabla\left(\Delta^{-1}X\right)\right] \cdot \nabla\delta_i + \frac{1}{a}\left[2\varphi\nabla \cdot (\mathbf{u}_i - \mathbf{u}) - (\mathbf{u}_i - \mathbf{u}) \cdot \nabla\varphi\right]$$

$$+ \frac{2}{a}\varphi\left[\delta_i\nabla \cdot (\mathbf{u}_i - \mathbf{u}) + 2(\mathbf{u}_i - \mathbf{u}) \cdot \nabla\varphi - 2\varphi\nabla \cdot (\mathbf{u}_i - \mathbf{u})\right] - \frac{1}{a}\delta_i(\mathbf{u}_i - \mathbf{u}) \cdot \nabla\varphi, \tag{106}$$

$$\frac{1}{a}\nabla \cdot (\dot{\mathbf{u}}_i + H\mathbf{u}_i) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u}_i \cdot \nabla\mathbf{u}_i) - \frac{\Delta}{a^2}\left[\mathbf{u}_i \cdot \nabla\left(\Delta^{-1}X\right)\right] + \frac{1}{a^2}\left(\mathbf{u} \cdot \nabla X + \frac{2}{3}X\nabla \cdot \mathbf{u}\right)$$

$$-\frac{2}{3a^2}\varphi\mathbf{u} \cdot \nabla\left(\nabla \cdot \mathbf{u}\right) + \frac{4}{a^2}\nabla \cdot \left[\varphi\left(\mathbf{u} \cdot \nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla \cdot \mathbf{u}\right)\right] + 2\frac{\Delta}{a^2}\left[\varphi\mathbf{u} \cdot (\mathbf{u}_i - \mathbf{u})\right]. \tag{107}$$

Equations (117),(118) coincide with Eqs. (39),(40) in [4]. Equation (93) gives

$$\frac{1}{a}\left[a\left(\mathbf{u}_i-\mathbf{u}\right)\right]^{\cdot}=-\frac{1}{a}\nabla\left\{(1-2\varphi)\left(\mathbf{u}_i-\mathbf{u}\right)\cdot\left[\mathbf{u}+\nabla\left(\Delta^{-1}X\right)\right]\right\}. \tag{108}$$

Equation (94),(95) give

$$\frac{1}{a^2}\left\{a^2\dot{\delta}+a\nabla\cdot(\delta\mathbf{u})-a\left[2\varphi\mathbf{u}-\nabla\left(\Delta^{-1}X\right)\right]\cdot\nabla\delta\right\}^{\cdot}-4\pi G\varrho\delta=\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})+\frac{\Delta}{a^2}\left[\mathbf{u}\cdot\nabla\left(\Delta^{-1}X\right)\right]$$
$$-\frac{1}{a^2}\left(\mathbf{u}\cdot\nabla X+\frac{2}{3}X\nabla\cdot\mathbf{u}\right)+\frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla\left(\nabla\cdot\mathbf{u}\right)-\frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u}-\frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right], \tag{109}$$

$$\frac{1}{a^2}\left\{a^2\dot{\delta}_i+a\nabla\cdot(\delta_i\mathbf{u}_i)-a\left[2\varphi\mathbf{u}_i-\nabla\left(\Delta^{-1}X\right)\right]\cdot\nabla\delta_i\right\}^{\cdot}-4\pi G\varrho\delta-\frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i)-\frac{\Delta}{a^2}\left[\mathbf{u}_i\cdot\nabla\left(\Delta^{-1}X\right)\right]$$
$$+\frac{1}{a^2}\left(\mathbf{u}\cdot\nabla X+\frac{2}{3}X\nabla\cdot\mathbf{u}\right)-\frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla\left(\nabla\cdot\mathbf{u}\right)+\frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u}-\frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]$$
$$=\frac{1}{a}\left[2\dot{\varphi}\nabla\cdot\left(\mathbf{u}_i-\mathbf{u}\right)-\left(\mathbf{u}_i-\mathbf{u}\right)\cdot\nabla\dot{\varphi}\right]-2\frac{\Delta}{a^2}\left[\varphi\mathbf{u}\cdot\left(\mathbf{u}_i-\mathbf{u}\right)\right]$$
$$-\frac{2}{a^2}\varphi\left\{\Delta\left[\mathbf{u}\cdot\left(\mathbf{u}_i-\mathbf{u}\right)\right]+\left(\nabla\cdot\mathbf{u}_i\right)\nabla\cdot\left(\mathbf{u}_i-\mathbf{u}\right)\right\}+\frac{1}{a^2}\left(\nabla\varphi\right)\cdot\left\{\nabla\left[\mathbf{u}\cdot\left(\mathbf{u}_i-\mathbf{u}\right)\right]+\left(\nabla\cdot\mathbf{u}_i\right)\left(\mathbf{u}_i-\mathbf{u}\right)\right\}. \tag{110}$$

Equation (99) becomes

$$\dot{\varphi}=\frac{1}{3a}\left[-X+2\varphi\nabla\cdot\mathbf{u}-\mathbf{u}\cdot\nabla\varphi\right]=-\frac{1}{2}\Delta^{-1}\nabla\left[\frac{1}{a}\mathbf{u}\cdot\nabla\left(\nabla\varphi\right)+\mathbf{u}\frac{\Delta}{a}\varphi\right]. \tag{111}$$

A close examination of above equations reveals the followings.

(i) Equations (104),(105),(109) are the same as Eqs. (25),(26),(28) in [4] which are valid in a single component fluid; we note, however, that $\mathcal{O}(|v_{(i)\alpha}|^2)$ correction terms, which simply decay in an expanding phase, appear even in the second-order perturbations, see Eqs. (66),(67),(70) above, or [5].

(ii) If we ignore the i-indices, Eqs. (106),(107),(110) are identical to Eqs. (104),(105),(109), respectively; we note, however, that the presence of $\mathcal{O}(|v_{(i)\alpha}|^2)$ correction terms causes differences even in the second-order perturbations, see Eqs. (50)-(55).

(iii) We already showed that, to the second order, even in the case of multi-component, the general relativistic equations are identical to the Newtonian ones, thus having relativistic/Newtonian correspondence. The presence of $\mathcal{O}(|v_{(i)\alpha}|^2)$ correction terms may cause differences, but we have shown that these corrections are simply decaying in the expanding phase.

(iv) The pure third-order correction terms in the above equations all involve φ-term to the linear order in various forms of convolution with the second-order terms. As we took the comoving gauge $v\equiv 0$, the spatial curvature variable φ is the same as a gauge-invariant combination $\varphi_v\equiv\varphi-aHv/c$ to the linear order.

(v) We note that, to the linear order, φ_v is one of the well known conserved quantity in the large-scale [20, 21]. For $K=0$, but considering general Λ, we have $\dot{\varphi}_v=0$, thus

$$\varphi_v=C(\mathbf{x}), \tag{112}$$

with vanishing decaying mode (in an expanding phase) to the leading order in the large-scale expansion [18].

(vi) We also note that the value of φ_v in the large-scale is of the same order as the gravitational potential fluctuation φ_χ which is again of the same order as the relative temperature fluctuations $\delta T/T$ of the cosmic microwave background radiation (CMB). In general we have [4]

$$\varphi_v=\varphi_\chi-aHv_\chi/c=-\delta\Phi/c^2+\dot{a}\Delta^{-1}\nabla\cdot\mathbf{u}/c^2, \tag{113}$$

to the linear order, where we have $\varphi_\chi=-\delta\Phi/c^2$ and $\mathbf{u}=-c\nabla v_\chi$; $\delta\Phi$ is the Newtonian gravitational potential identified to the linear order in [14, 18, 22]. For $K=0=\Lambda$ we have

$$\varphi_v=\frac{5}{3}\varphi_\chi. \tag{114}$$

The temperature anisotropy of CMB, in a flat background without the cosmological constant, gives [23, 24]

$$\frac{\delta T}{T}\sim\frac{1}{3}\varphi_\chi=\frac{1}{3}\frac{\delta\Phi}{c^2}\sim\frac{1}{5}\varphi_v\sim\frac{1}{5}C, \tag{115}$$

to the linear order; this is a part of the Sachs-Wolfe effect [23]. The COBE observations of CMB give $\delta T/T \sim 10^{-5}$ [25], thus

$$\varphi_v \sim 5 \times 10^{-5}, \tag{116}$$

in the large-scale limit near horizon scale. Therefore, the pure general relativistic third-order correction terms are independent of the presence of the horizon scale and are smaller by factor 5×10^{-5} compared with the second-order relativistic/Newtonian terms due to the low level anisotropies of the cosmic microwave background radiation.

B. Contributions from tensor-type perturbations

Here we present set of equations describing the scalar-type perturbation equations to the third order, now including the contributions from the tensor-type perturbations. We continue to use the Newtonian variables identified in Eq. (101).

Equations (89)-(92) become

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta \mathbf{u}) + \frac{1}{a}\left(2\varphi\mathbf{u} - \nabla\Delta^{-1}X\right) \cdot \nabla\delta + \frac{2}{a}\delta^{,\alpha}u^\beta C^{(t)}_{\alpha\beta}, \tag{117}$$

$$\frac{1}{a}\nabla \cdot (\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right)$$

$$-\frac{\Delta}{a^2}\left[\mathbf{u} \cdot \nabla\left(\Delta^{-1}X\right)\right] + \frac{1}{a^2}\left(\mathbf{u} \cdot \nabla X + \frac{2}{3}X\nabla \cdot \mathbf{u}\right) - \frac{2}{3a^2}\varphi\mathbf{u} \cdot \nabla\left(\nabla \cdot \mathbf{u}\right) + \frac{4}{a^2}\nabla \cdot \left[\varphi\left(\mathbf{u} \cdot \nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla \cdot \mathbf{u}\right)\right]$$

$$+\frac{2}{a^2}u^\alpha\nabla^\beta\left(\nabla \cdot \mathbf{u}\right)C^{(t)}_{\alpha\beta} + 2\dot{C}^{(t)\alpha\beta}\left[\frac{4}{a}\varphi u_{\alpha|\beta} + \frac{2}{a}u_\alpha\nabla_\beta\varphi + 2\varphi\dot{C}^{(t)}_{\alpha\beta} - \frac{1}{a}\left(\Delta^{-1}X\right)_{,\alpha|\beta}\right]$$

$$+2\left(\frac{1}{a}u^{\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right)\left[-\frac{2}{3a}\left(\nabla \cdot \mathbf{u}\right)C^{(t)}_{\alpha\beta} + \frac{1}{a}u^\gamma\left(2C^{(t)}_{\gamma\alpha|\beta} - C^{(t)}_{\alpha\beta|\gamma}\right) + 2C^{(t)}_{\beta\gamma}\left(\frac{1}{a}u^\gamma{}_{|\alpha} + \dot{C}^{(t)\gamma}{}_\alpha\right)\right], \tag{118}$$

$$\dot{\delta}_i + \frac{1}{a}\nabla \cdot \mathbf{u}_i = -\frac{1}{a}\nabla \cdot (\delta_i\mathbf{u}_i) + \frac{1}{a}\left[2\varphi\mathbf{u}_i - \nabla\left(\Delta^{-1}X\right)\right] \cdot \nabla\delta_i + \frac{2}{a}\delta_i^{,\alpha}u_i^\beta C^{(t)}_{\alpha\beta}$$

$$+\frac{1}{a}\left[2\varphi\nabla \cdot \left(\mathbf{u}_i - \mathbf{u}\right) - \left(\mathbf{u}_i - \mathbf{u}\right) \cdot \nabla\varphi + 2C^{(t)}_{\alpha\beta}\left(u_i^\alpha - u^\alpha\right)^{|\beta}\right]$$

$$+\frac{2}{a}\varphi\left[\delta_i\nabla \cdot \left(\mathbf{u}_i - \mathbf{u}\right) + 2\left(\mathbf{u}_i - \mathbf{u}\right) \cdot \nabla\varphi - 2\varphi\nabla \cdot \left(\mathbf{u}_i - \mathbf{u}\right)\right] - \frac{1}{a}\delta_i\left(\mathbf{u}_i - \mathbf{u}\right) \cdot \nabla\varphi$$

$$+\frac{2}{a}C^{(t)\alpha\beta}\left\{\left(\delta_i - 4\varphi\right)\left(u_{i\alpha} - u_\alpha\right)_{|\beta} - \varphi_{,\alpha}\left(u_{i\beta} - u_\beta\right) + C^{(t)}_{\alpha\beta|\gamma}\left(u_i^\gamma - u^\gamma\right) - 2\left[C^{(t)}_{\beta\gamma}\left(u_i^\gamma - u^\gamma\right)\right]_{|\alpha}\right\}, \tag{119}$$

$$\frac{1}{a}\nabla \cdot (\dot{\mathbf{u}}_i + H\mathbf{u}_i) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u}_i \cdot \nabla\mathbf{u}_i) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right)$$

$$-\frac{\Delta}{a^2}\left[\mathbf{u}_i \cdot \nabla\left(\Delta^{-1}X\right)\right] + \frac{1}{a^2}\left(\mathbf{u} \cdot \nabla X + \frac{2}{3}X\nabla \cdot \mathbf{u}\right)$$

$$-\frac{2}{3a^2}\varphi\mathbf{u} \cdot \nabla\left(\nabla \cdot \mathbf{u}\right) + \frac{4}{a^2}\nabla \cdot \left[\varphi\left(\mathbf{u} \cdot \nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla \cdot \mathbf{u}\right)\right] + 2\frac{\Delta}{a^2}\left[\varphi\mathbf{u} \cdot \left(\mathbf{u}_i - \mathbf{u}\right) + u^\alpha\left(u_i^\beta - u^\beta\right)C^{(t)}_{\alpha\beta}\right]$$

$$+\frac{2}{a^2}u^\alpha\nabla^\beta\left(\nabla \cdot \mathbf{u}\right)C^{(t)}_{\alpha\beta} + 2\dot{C}^{(t)\alpha\beta}\left[\frac{4}{a}\varphi u_{\alpha|\beta} + \frac{2}{a}u_\alpha\nabla_\beta\varphi + 2\varphi\dot{C}^{(t)}_{\alpha\beta} - \frac{1}{a}\left(\Delta^{-1}X\right)_{,\alpha|\beta}\right]$$

$$+2\left(\frac{1}{a}u^{\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right)\left[-\frac{2}{3a}\left(\nabla \cdot \mathbf{u}\right)C^{(t)}_{\alpha\beta} + \frac{1}{a}u^\gamma\left(2C^{(t)}_{\gamma\alpha|\beta} - C^{(t)}_{\alpha\beta|\gamma}\right) + 2C^{(t)}_{\beta\gamma}\left(\frac{1}{a}u^{|\gamma}{}_\alpha + \dot{C}^{(t)\gamma}{}_\alpha\right)\right]. \tag{120}$$

Equations (117),(118) coincide with Eqs. (39),(40) in [4]. Equation (93) gives

$$\frac{1}{a}\left[a\left(\mathbf{u}_i - \mathbf{u}\right)\right]^\cdot = -\frac{1}{a}\nabla\left\{\left(1 - 2\varphi\right)\left(\mathbf{u}_i - \mathbf{u}\right) \cdot \left[\mathbf{u} + \nabla\left(\Delta^{-1}X\right)\right] - 2u^\alpha\left(u_i^\beta - u^\beta\right)C^{(t)}_{\alpha\beta}\right\}. \tag{121}$$

Equation (93) gives

$$
\frac{1}{a^2}\left\{a^2\dot\delta + a\nabla\cdot(\delta\mathbf{u}) - a\left[2\varphi\mathbf{u} - \nabla\left(\Delta^{-1}X\right)\right]\cdot\nabla\delta - 2a\delta^{,\alpha}u^\beta C^{(t)}_{\alpha\beta}\right\}^{\cdot} - 4\pi G\varrho\delta
$$

$$
= \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot C^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot C^{(t)}_{\alpha\beta}\right) + \frac{\Delta}{a^2}\left[\mathbf{u}\cdot\nabla\left(\Delta^{-1}X\right)\right] - \frac{1}{a^2}\left(\mathbf{u}\cdot\nabla X + \frac{2}{3}X\nabla\cdot\mathbf{u}\right)
$$

$$
+ \frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla\left(\nabla\cdot\mathbf{u}\right) - \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]
$$

$$
- \frac{2}{a^2}u^\alpha\nabla^\beta\left(\nabla\cdot\mathbf{u}\right)C^{(t)}_{\alpha\beta} - 2\dot C^{(t)\alpha\beta}\left[\frac{4}{a}\varphi u_{\alpha|\beta} + \frac{2}{a}u_\alpha\nabla_\beta\varphi + 2\varphi\dot C^{(t)}_{\alpha\beta} - \frac{1}{a}\left(\Delta^{-1}X\right)_{,\alpha|\beta}\right]
$$

$$
- 2\left(\frac{1}{a}u^{\alpha|\beta} + \dot C^{(t)\alpha\beta}\right)\left[-\frac{2}{3a}\left(\nabla\cdot\mathbf{u}\right)C^{(t)}_{\alpha\beta} + \frac{1}{a}u^\gamma\left(2C^{(t)}_{\gamma\alpha|\beta} - C^{(t)}_{\alpha\beta|\gamma}\right) + 2C^{(t)}_{\beta\gamma}\left(\frac{1}{a}u^\gamma{}_{|\alpha} + \dot C^{(t)\gamma}{}_\alpha\right)\right], \qquad (122)
$$

$$
\frac{1}{a^2}\left\{a^2\dot\delta_i + a\nabla\cdot(\delta_i\mathbf{u}_i) - a\left[2\varphi\mathbf{u}_i - \nabla\left(\Delta^{-1}X\right)\right]\cdot\nabla\delta_i - 2a\delta_i^{,\alpha}u_i^\beta C^{(t)}_{\alpha\beta}\right\}^{\cdot} - 4\pi G\varrho\delta
$$

$$
- \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) - \dot C^{(t)\alpha\beta}\left(\frac{2}{a}u_{i\alpha|\beta} + \dot C^{(t)}_{\alpha\beta}\right) - \frac{\Delta}{a^2}\left[\mathbf{u}_i\cdot\nabla\left(\Delta^{-1}X\right)\right] + \frac{1}{a^2}\left(\mathbf{u}\cdot\nabla X + \frac{2}{3}X\nabla\cdot\mathbf{u}\right)
$$

$$
- \frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla\left(\nabla\cdot\mathbf{u}\right) + \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]
$$

$$
+ \frac{2}{a^2}u^\alpha\nabla^\beta\left(\nabla\cdot\mathbf{u}\right)C^{(t)}_{\alpha\beta} + 2\dot C^{(t)\alpha\beta}\left[\frac{4}{a}\varphi u_{\alpha|\beta} + \frac{2}{a}u_\alpha\nabla_\beta\varphi + 2\varphi\dot C^{(t)}_{\alpha\beta} - \frac{1}{a}\left(\Delta^{-1}X\right)_{,\alpha|\beta}\right]
$$

$$
+ 2\left(\frac{1}{a}u^{\alpha|\beta} + \dot C^{(t)\alpha\beta}\right)\left[-\frac{2}{3a}\left(\nabla\cdot\mathbf{u}\right)C^{(t)}_{\alpha\beta} + \frac{1}{a}u^\gamma\left(2C^{(t)}_{\gamma\alpha|\beta} - C^{(t)}_{\alpha\beta|\gamma}\right) + 2C^{(t)}_{\beta\gamma}\left(\frac{1}{a}u^{|\gamma}{}_\alpha + \dot C^{(t)\gamma}{}_\alpha\right)\right]
$$

$$
= \frac{1}{a}\left[2\dot\varphi\nabla\cdot(\mathbf{u}_i - \mathbf{u}) - (\mathbf{u}_i - \mathbf{u})\cdot\nabla\dot\varphi\right] - 2\frac{\Delta}{a^2}\left[\varphi\mathbf{u}\cdot(\mathbf{u}_i - \mathbf{u}) + u^\alpha\left(u_i^\beta - u^\beta\right)C^{(t)}_{\alpha\beta}\right]
$$

$$
- \frac{2}{a^2}\varphi\left\{\Delta\left[\mathbf{u}\cdot(\mathbf{u}_i - \mathbf{u})\right] + (\nabla\cdot\mathbf{u}_i)\nabla\cdot(\mathbf{u}_i - \mathbf{u})\right\} + \frac{1}{a^2}\left(\nabla\varphi\right)\cdot\left\{\nabla\left[\mathbf{u}\cdot(\mathbf{u}_i - \mathbf{u})\right] + (\nabla\cdot\mathbf{u}_i)(\mathbf{u}_i - \mathbf{u})\right\}
$$

$$
- \frac{2}{a^2}C^{(t)\alpha\beta}\left\{\left[\mathbf{u}\cdot(\mathbf{u}_i - \mathbf{u})\right]_{,\alpha|\beta} + (\nabla\cdot\mathbf{u}_i)(u_{i\alpha} - u_\alpha)_{|\beta} - a\dot C^{(t)}_{\alpha\beta|\gamma}\left(u_i^\gamma - u^\gamma\right) + 2a\left[\dot C^{(t)}_{\beta\gamma}\left(u_i^\gamma - u^\gamma\right)\right]_{|\alpha}\right\}
$$

$$
+ \frac{2}{a}\dot C^{(t)\alpha\beta}\left\{(\delta_i - 4\varphi)(u_{i\alpha} - u_\alpha)_{|\beta} - \varphi_{,\alpha}(u_{i\beta} - u_\beta) + C^{(t)}_{\alpha\beta|\gamma}\left(u_i^\gamma - u^\gamma\right) - 2\left[C^{(t)}_{\beta\gamma}\left(u_i^\gamma - u^\gamma\right)\right]_{|\alpha}\right\}. \qquad (123)
$$

Equations (98),(99) become

$$
X \equiv 2\varphi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi + C^{(t)\alpha\beta}\left(2u_{\alpha|\beta} - a\dot C^{(t)}_{\alpha\beta}\right)
$$

$$
+ \frac{3}{2}\Delta^{-1}\nabla^\alpha\left[\mathbf{u}\cdot\nabla\left(\nabla_\alpha\varphi\right) + u_\alpha\Delta\varphi + u^\beta\Delta C^{(t)}_{\alpha\beta} - a\varphi^{;\beta}\dot C^{(t)}_{\alpha\beta} + 2aC^{(t)\beta\gamma}\dot C^{(t)}_{\alpha\beta|\gamma} + aC^{t)}_{\beta\gamma|\alpha}\dot C^{(t)\beta\gamma}\right], \qquad (124)
$$

$$
\dot\varphi = \frac{1}{3a}\left[-X + 2\varphi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi + 2C^{(t)\alpha\beta}\left(u_{\alpha|\beta} + a\dot C^{(t)}_{\alpha\beta}\right)\right]
$$

$$
= C^{(t)\alpha\beta}\dot C^{(t)}_{\alpha\beta} - \frac{1}{2}\Delta^{-1}\nabla^\alpha\left[\frac{1}{a}\mathbf{u}\cdot\nabla\left(\nabla_\alpha\varphi\right) + u_\alpha\frac{\Delta}{a}\varphi + u^\beta\frac{\Delta}{a}C^{(t)}_{\alpha\beta} - \varphi^{;\beta}\dot C^{(t)}_{\alpha\beta} + 2C^{(t)\beta\gamma}\dot C^{(t)}_{\alpha\beta|\gamma} + C^{t)}_{\beta\gamma|\alpha}\dot C^{(t)\beta\gamma}\right] (125)
$$

V. DISCUSSION

In this work we have successfully derived pure general relativistic correction terms appearing in the third-order perturbations of the zero-pressure irrotational multi-component fluids in a flat background. We have ignored $\mathcal{O}(|\mathbf{u} - \mathbf{u}_i|^2)$ correction terms which simply decay in an expanding phase. Our main results are presented in Eqs. (101)-(111), and in Eqs. (117)-(125) in the presence of the gravitational waves. The equations for the collective component are identical to the ones in the single component case. If we further ignore $\mathcal{O}(\mathbf{u} - \mathbf{u}_i)$ correction terms, which are again simply decaying in an expanding phase, the forms of relativistic correction terms in the individual component are the same as the ones in the collective component. Our results show that, even in the multi-component situation, the pure relativistic correction terms are smaller by a factor

$$
\varphi_v \sim \frac{5}{3}\frac{\delta\Phi}{c^2} \sim 5\frac{\delta T}{T} \sim 5\times 10^{-5}, \qquad (126)
$$

compared with the relativistic/Newtonian second-order terms and are independent of the presence of horizon scale.

By taking different temporal gauge condition (hypersurface condition) we can easily introduce apparently horizon dependent terms with arbitrarily huge amplitudes. The exact relativistic/Newtonian correspondence to the second-order perturbations was available essentially due to our proper choice of gauge conditions and correct identifications of relativistic variables with the Newtonian ones. In our third-order extension we have assumed the same identification holds even in the third-order perturbations, which might not be necessarily the unique choice. However, the properties (the smallness of the amplitudes and independence from the horizon scale) of our third-order correction terms assure that our choice of the gauge and identifications are very likely to be the correct and best ones even to the third order.

Our results may have practically important implications in currently favored cosmological pursuits by assuring the use of Newtonian physics in the large-scale nonlinear processes which often involve two-component zero-pressure fluids (say, dust and cold dark matter). As we have shown the exact relativistic/Newtonian correspondence to the second order and small horizon-independent third-order correction terms, it is now more secure to use Newtonian physics near (and even beyond) the horizon scale which is indeed a noticeable trend in current cosmological simulations [26]. At this point it might be also worth emphasizing the effects of rotational perturbations. In [5] we have shown that the rotational perturbations to the second-order also have relativistic/Newtonian correspondence in the small-scale (sub-horizon scale) limit. As the numerical simulations naturally include the rotational mode, this might be another good news to the cosmology community based on Newtonian physics.

Although we have estimated that the third-order pure general relativistic correction terms are small in the current large-scale structures, it would be still interesting to see whether cosmology could reach a stage where $\varphi_v \sim 5 \times 10^{-5}$-factor smaller correction terms could have noticeable effect on the large-scale structure formation processes. In either case the third-order terms we have presented in this work are the first non-vanishing general relativistic correction terms in Newtonian nonlinear equations, and may have historical as well as practical importance in cosmology. Analytic studies in the Newtonian context of a single component zero-pressure, irrotational fluid have been actively investigated in [27]. Whether our general relativistic corrections appearing in the third order (both in the single-component and multi-component cases), and corrections appearing even in the second order by effects of pressure, rotation, and background curvature, will have noticeable roles in the large-scale structure formation process are left for future investigations.

Acknowledgments

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[1] H. Noh and J. Hwang, Phys. Rev. D **69**, 104011 (2004), Preprint astro-ph/0305123.

[2] H. Noh and J. Hwang, Class. Quant. Grav. **22**, 3181 (2005), Preprint gr-qc/0412127; J. Hwang and H. Noh, Phys. Rev. D **72**, 044011 (2005), Preprint gr-qc/0412128; J. Hwang and H. Noh, Monthly Not. R. Astron. Soc. **367**, 1515 (2006), Preprint astro-ph/0507159; H. Noh and J. Hwang, Gen. Rel. Grav. **38**, 703 (2006), Preprint astro-ph/0512636.

[3] J. Hwang and H. Noh, Phys. Rev. D **73**, 044021 (2006), Preprint astro-ph/0601041.

[4] J. Hwang and H. Noh, Phys. Rev. D **72**, 044012 (2005), Preprint gr-qc/0412129.

[5] J. Hwang, and H. Noh, Phys. Rev. D, submitted, Preprint astro-ph/0704.1927.

[6] A.A. Friedmann, Zeitschrift für Physik **10**, 377 (1922), and *ibid.* **21**, 326 (1924); both papers are translated in *Cosmological-constants: papers in modern cosmology*, edited by J. Bernstein and G. Feinberg (Columbia Univ. Press, New York, 1986), p49 and p59; H.P. Robertson, Proceedings of the National Academy of Science **15**, 822 (1929).

[7] E.A. Milne, Quart. J. Math. **5**, 64 (1934); W.H. McCrea and E.A. Milne, Quart. J. Math. **5**, 73 (1934).

[8] D. Layzer, Astron. J. **59**, 268 (1954); D.S. Lemons, Am. J. Phys. **56**, 502 (1988).

[9] E.M. Lifshitz, J. Phys. (USSR) **10**, 116 (1946); E.M. Lifshitz and I.M. Khalatnikov, Adv. Phys. **12**, 185 (1963).

[10] W.B. Bonnor, Mon. Not. R. Astron. Soc. **117**, 104 (1957).

[11] P.J.E. Peebles, *The large-scale structure of the universe*, (Princeton Univ. Press, Princeton, 1980).

[12] J. Ehlers, Proceedings of the mathematical-natural science of the Mainz academy of science and literature, Nr. **11**, 792 (1961), translated in Gen. Rel. Grav. **25**, 1225 (1993); G.F.R. Ellis, in *General relativity and cosmology, Proceedings of the international summer school of physics Enrico Fermi course 47*, edited by R. K. Sachs (Academic Press, New York, 1971), p104; in *Cargese Lectures in Physics*, edited by E. Schatzmann (Gorden and Breach, New York, 1973), p1.

[13] R. Arnowitt, S. Deser, and C.W. Misner, in *Gravitation: an introduction to current research*, edited by L. Witten (Wiley, New York, 1962) p. 227.

[14] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[15] J. Hwang and E.T. Vishniac, Astrophys. J. **353**, 1 (1990).

[16] J.M. Bardeen, *Particle Physics and Cosmology*, edited by L. Fang and A. Zee (Gordon and Breach, London, 1988) p1.

[17] J. Hwang and H. Noh, Class. Quant. Grav. **19**, 527 (2002) Preprint astro-ph/0103244.

[18] J. Hwang and H. Noh, Gen. Rel. Grav. **31**, 1131 (1999), Preprint astro-ph/9907063.

[19] H. Nariai, Prog. Theor. Phys. **41**, 686 (1969).

[20] J. Hwang, Astrophys. J. **375**, 443 (1991).

[21] J. Hwang and H. Noh, Phys. Rev. D **71**, 063536 (2005), Preprint gr-qc/0412126 .

[22] E.R. Harrison, Rev. Mod. Phys. **39**, 862 (1967).

[23] R.K. Sachs and A.M. Wolfe, Astrophys. J. **147**, 73 (1967).

[24] J. Hwang and H. Noh, Phys. Rev. D **59**, 067302 (1999), Preprint astro-ph/9812007.

[25] G.F. Smoot, *et al.* Astrophys. J. **396**, L1 (1992).

[26] J.M. Colberg, et al., Monthly Not. R. Astron. Soc. **319**, 209 (2000), Preprint astro-ph/0005259; A. Jenkins, et al., Monthly Not. R. Astron. Soc. **321**, 372 (2001), Preprint astro-ph/0005260; A.E. Evrard, et al., Astrophys. J. **573**, 7 (2002), Preprint astro-ph/0110246; P. Bode and J.P. Ostriker, Astrophys. J. Suppl. **145**, 1 (2003), Preprint astro-ph/0302065; J. Dubinski, J. Kim, C. Park R. Humble, New Astron., **9**, 111 (2003), Preprint astro-ph/0304467; V. Springel, et al., Nature **435**, 629 (2005), Preprint astro-ph/0504097; V. Springel, C.S. Frenk, S.D.M. White, Nature **440**, 1137 (2006), Preprint astro-ph/0604561.

[27] E.T. Vishniac, Monthly Not. R. Astron. Soc. **203**, 345 (1983); M.H. Goroff, B. Grinstein, S.-J. Rey, and M.B. Wise, Astrophys. J., **311**, 6 (1986); F. Bernardeau, S. Colombi, E. Gaztanaga, and R. Scoccimarro, Phys. Rep. **367**, 1 (2002), Preprint astro-ph/0112551.